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Existence and uniqueness of solutions to superdifferential equations

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We state and prove the theorem of existence and uniqueness of solutions to ordinary superdifferential equations on supermanifolds. It is shown that any supervector field, $X = X_0 + X_1$, has a unique integral flow, $\Gamma\colon \mathbb{R}^{1|1} \times (M, \mathcal{A}_M) \to (M, \mathcal{A}_M)$, satisfying a given initial condition. A necessary and sufficient condition for this integral flow to yield an $\mathbb{R}^{1|1}$-action is obtained: the homogeneous components, X_0, and, X_1, of the given field must define a Lie superalgebra of dimension $(1,1)$. The supergroup structure on $\mathbb{R}^{1|1}$, however, has to be specified: there are three non-isomorphic Lie supergroup structures on $\mathbb{R}^{1|1}$, all of which have addition as the group operation in the underlying Lie group $\mathbb{R}$. On the other extreme, even if X_0, and X_1 do not close to form a Lie superalgebra, the integral flow of X is uniquely determined and is independent of the Lie supergroup structure imposed on $\mathbb{R}^{1|1}$. This fact makes it possible to establish an unambiguous relationship between the algebraic Lie derivative of supergeometric objects (e.g., superforms), and its geometrical definition in terms of integral flows. It is shown by means of examples that if a supergroup structure in $\mathbb{R}^{1|1}$ is fixed, some flows obtained from left-invariant supervector fields on Lie supergroups may fail to define an $\mathbb{R}^{1|1}$-action of the chosen structure. Finally, necessary and sufficient conditions for the integral flows of two supervector fields to commute are given.

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1. Introduction

We prove here the theorem of existence and uniqueness of solutions to superdifferential equations on supermanifolds. This work is based on two previous approaches—each one followed by each of the authors separately (refs. [7] and [9], respectively). Both predecessor papers dealt with the problem of integrating supervector fields on supermanifolds, but the results reached by each one of them were only partial. In ref. [7], a unique way of integrating even supervector fields was obtained, but ad hoc techniques were required for the odd ones. Even so, not all of them could have an integral in the sense defined there; integral flows in ref. [7] depended only on one real parameter $t \in \mathbb{R}$. On the other hand, the approach in ref. [9] provided a better way of making sense of the ordinary differential equation defined by any supervector field. This was achieved by introducing $\mathbb{R}^{1|1}$ as the parameter superspace to carry out the integration, and by using the evaluation morphism on points to completely determine the C^∞ functions that build up the flow. The proof of the theorem on existence and uniqueness of solutions there, was based on the ideas of the pioneering work of Shander [13]: To determine first the normal forms for the superfields so as to actually carry out the integration on the simplest coordinate version of each. The normal-form problem, however, was not solved in ref. [9], and the theorem was therefore proved only for a subclass of supervector fields: those having a normal form in $\mathbb{R}^{1|1}$. Nevertheless, these included the known examples in the literature so far, and provided some new ones (cf. refs. [5,2,13]).

We are now very pleased to communicate in this paper the best statement of the theorem, and its most unrestricted proof: i.e., one without any regard on parities, normal forms, special integrating parameters or techniques, etc. (cf. theorem 3.5, below). But before giving the details, it is pertinent to make some comments about the nature of the problem, the nature of our approach, and the significance of the results.

First of all, the problem of posing ordinary differential equations on supermanifolds (or, on any similar geometric category), combines the supervector field to be integrated, X, with the solution of the equation, Γ, so as to Γ-relate X with some fixed derivation D in the integrating parameter superspace, $\mathcal{T}$. We call the pair $(\mathcal{T}, D)$ the *integrating model* for the equation. Thus, the starting point is always the equation,

$$D \circ \Gamma^* = \Gamma^* \circ X, \tag{1.1}$$

(cf. section 2 below for precise definitions and statements). Since Γ^* is a map of superalgebras, it preserves the $\mathbb{Z}_2$-grading, and therefore, eq. (1.1) may immediately be split into two equations; namely,

$$D_0 \circ \Gamma^* = \Gamma^* \circ X_0 \quad \text{and} \quad D_1 \circ \Gamma^* = \Gamma^* \circ X_1, \tag{1.2}$$

where, $X = X_0 + X_1$, and $D = D_0 + D_1$, are the corresponding $\mathbb{Z}_2$-decompositions of X, and D. In particular, to integrate odd superfields one requires at least a non-zero D_1. On the other hand, since the supercommutators of two Γ-related derivations are again Γ-related, it follows from (1.2) that

$$[D_0, D_1] \circ \Gamma^* = \Gamma^* \circ [X_0, X_1] \quad \text{and} \quad [D_1, D_1] \circ \Gamma^* = \Gamma^* \circ [X_1, X_1]. \tag{1.3}$$

These relations may produce some non-trivial conditions on the superfields X to be integrated depending on the values of $[D_0, D_1]$, and $[D_1, D_1]$. It is natural to assume that the integrating parameter superspace $\mathcal{T}$ is a Lie supergroup, and that D_0, and D_1 are left-invariant supervector fields, so that they form a $(1,1)$-dimensional Lie superalgebra. If this is the case, there are *real constants* a, and b (and in fact, $a\,b = 0$), such that

$$[D_0, D_1] = a\,D_1 \quad \text{and} \quad [D_1, D_1] = b\,D_0. \tag{1.4}$$

In particular, if Γ^* happens to be monic (cf. ref. [1]), (1.1) becomes a well-posed equation only for those superfields satisfying the "integrability conditions",

$$[X_0, X_1] = a\,X_1 \quad \text{and} \quad [X_1, X_1] = b\,X_0. \tag{1.5}$$

There are, however, various reasons to pursuit the idea that any supervector field must be integrable, in the sense of giving rise to an *integral flow*, $\Gamma: \mathcal{T} \times (M, \mathcal{A}_M) \to (M, \mathcal{A}_M)$. In fact, an integral flow for *any superfield* is needed in order to relate the dynamical (geometrical) definition of the Lie derivative of any supergeometrical object (e.g., superdifferential forms), to its corresponding algebraic characterization. The latter usually makes good sense, no matter what superfield is chosen to take the derivative along to. The best example at hand is this: the Lie derivative of any superform ω, may be defined algebraically by

$$\mathcal{L}_X\omega = i(X) \circ \mathrm{d}\omega + \mathrm{d} \circ i(X)\omega, \tag{1.6}$$

without imposing conditions like (1.5) on X. One would like to understand this formula as the quantitative result of a geometrical assertion: the rate of change of ω along the flow generated by X. In particular, one would like to conclude that when the Lie derivative of something is zero, that something does not change along the flow. This is the crucial step in proving some geometrical assertions. To quote a concrete example, let us mention that this result is needed to show that the integral flow of a supervector field acts by supersymplectic transformations, if and only if it is superhamiltonian (see for example, refs. [10,12]; see also refs. [6,11] for the basics of supersymplectic supermanifolds).

The way to suppress the conditions on the homogeneous components of the field, and to produce a uniquely determined integral flow for any supervector field, is to pose the differential equation in terms of the evaluation morphism on points of the superparameter space $\mathcal{T}$. This was precisely the main contribution of ref. [9]. We recall that in the category of supermanifolds there is a unique terminal object: a single point with the constant structure sheaf $\mathbb{R}$. It is natural

Table 1

	$\mu((t_1,\tau_1),(t_2,\tau_2))$	D_0	D_1
Type 1	$(t_1+t_2,\tau_1+\tau_2)$	∂_t	∂_τ
Type 2	$(t_1+t_2+\tau_1\tau_2,\tau_1+\tau_2)$	∂_t	$\partial_\tau+\tau\partial_t$
Type 3	$(t_1+t_2,\tau_2+e^{t_2}\tau_1)$	$\partial_t+\tau\partial_\tau$	∂_τ

in terms of it to produce an evaluation morphism on points, and to make sense of, $\mathrm{ev}|_{t=t_0}$, as a morphism of superalgebras. Thus, the differential equation must be (cf. section 2 below for the precise definitions),

$$\mathrm{ev}|_{t=t_0}\circ D\circ \Gamma^* = \mathrm{ev}|_{t=t_0}\circ \Gamma^*\circ X. \tag{1.7}$$

What remains then is to select a specific integrating model, $(\mathcal{T},D)$. Now, the even part of any supervector field on $(M,\mathcal{A}_M)$ canonically projects onto a smooth vector field $\widetilde{X}$ on M whose integral flow always defines (locally, at least) an action of the additive group $\mathbb{R}$ on M. Therefore, it is only natural to require that $\mathcal{T}_{\mathrm{red}}=\mathbb{R}$, and the underlying smooth map $\widetilde{\mu}$ of the Lie supergroup operation μ: $\mathcal{T}\times\mathcal{T}\to\mathcal{T}$ be addition in $\mathbb{R}$. On the other hand, letting D_0, and D_1 be left invariant supervector fields, the possible choices are forced by the following procedure: First, determine all the $(1,1)$-dimensional Lie superalgebras over the reals. Then, look at their corresponding connected, simply connected, $(1,1)$-dimensional Lie supergroups having addition as their underlying operation in $\mathbb{R}$. Finally, realize D_0, and D_1 as left invariant supervector fields. It turns out that there are three different $(1,1)$-dimensional Lie supergroup structures on $\mathbb{R}^{1|1}$ extending addition on $\mathbb{R}$. We shall label these structures by the numbers 1, 2, and 3. Thus, if (1.4) is satisfied, the corresponding Lie supergroup structures on $\mathbb{R}^{1|1}$, and their left invariant supervector fields are given according to table 1.

(The Lie supergroup structure of type 2 is locally isomorphic to the supermultiplicative structure given by $\mu((t_1,\tau_1),(t_2,\tau_2)) = (t_1t_2+\tau_1\tau_2, t_1\tau_2+t_2\tau_1)$.) Note that $D=D_0+D_1$, is always of the form:

$$D=\partial_t+\partial_\tau+\tau D'.$$

Now, one can prove that the integral flow obtained when using D is exactly the same as that obtained when using $\partial_t+\partial_\tau$ (cf. section 2.3 below). In other words, for the sole purpose of determining the integral flow, Γ: $\mathbb{R}^{1|1}\times(M,\mathcal{A}_M)\to(M,\mathcal{A}_M)$, the detailed supergroup structure on $\mathbb{R}^{1|1}$ is irrelevant as long as the differential equation is posed as in (1.7). When it furthermore happens that the homogeneous components of the supervector field do form a $(1,1)$-dimensional Lie superalgebra, the differential equation may be posed as in (1.1) without the evaluation morphism, and a Lie supergroup action of the $\mathbb{R}^{1|1}$ supergroup corresponding to that Lie superalgebra is defined by the integral flow (cf. theorem 3.6 below).

Note that conditions (1.5) have nothing to do with the integrability criteria given by Frobenius theorem on the bundle that trivializes the particular supermanifold on which X is defined: eqs. (1.5) are stronger because a and b are *real constants*. When these conditions are not satisfied, all what happens is that the integral flow does not behave like the real one-parameter exponential of a C^∞ vector field. The integral flow exists, but it fails to define a Lie supergroup action of $\mathbb{R}^{1|1}$ on $(M, \mathcal{A}_M)$. This phenomenon might be exaggerated if a definite integrating model is fixed. For example, fixing the Type 1 supergroup structure as *the* integrating model $(\mathcal{T}, D)$, the integral flow obtained for a non-even left invariant supervector field on the Lie supergroup $\mathrm{GL}(1|1)$ (i.e., the multiplicative supergroup structure on $\mathbb{R}^{1|1}$) does not define a Lie supergroup action of $\mathcal{T}$. The reason is of course that the Type 1, and Type 2 structures are not isomorphic: The exponential morphism—understood as the "point" determined on the supergroup by flowing along the integral "curve" of a left invariant supervector field after a unit of "time" from some prescribed initial direction—does not provide a Lie supergroup homomorphism in this case.

The main results of this work (theorems 3.5, and 3.6) are presented with no commitment to any particular type of supergroup structure on $\mathbb{R}^{1|1}$. Our original approach made use of a specific choice (Type 1) arguing that for such a supergroup structure the correspondence that makes it possible to view an arbitrary section of the structural sheaf of a given supermanifold as a morphism from the supermanifold into $\mathbb{R}^{1|1}$ was addition preserving. However, M. Rothstein has pointed out to us that in so doing one leaves out some of the interesting Lie theoretic phenomena arising from the integration process. Besides, almost no new work had to be done in order to present the results in the more general setting because the actual computation of the integral flow Γ does not depend on the Lie supergroup structure of $\mathbb{R}^{1|1}$.

The paper is organized as follows: section 2 gives the basic definitions, and it is based on ref. [9]. The various Lie supergroup structures on $\mathbb{R}^{1|1}$ are given, and it is shown that for the actual determination of the flow only ∂_t, and ∂_τ may be used. Section 3 states the theorem of existence and uniqueness of solutions. Its proof is considerably reduced to the proof of the same theorem for an even superfield, X_0, but this is precisely the theorem proved in ref. [7], which we translate so as to fit with our general scheme here. Following refs. [10,12], we define in section 4 Lie derivatives of superforms in terms of our integral flows and show that one may compute these Lie derivatives algebraically with only interior multiplication, and exterior differentiation, as expected. Finally, section 5 provides the details for determining the left invariant supervector fields of the different Lie supergroup structures of $\mathbb{R}^{1|1}$. Needless to mention the relevance of having settled the integration question, as it is a fundamental tool in some applications. Concretely, we are thinking of some physical and geometrical considerations involving the Euler–Lagrange equations studied in ref. [8], and Hamilton's equations studied in refs. [10,12]. We also hope that the methods developed here will be of some help in the infinite dimensional theory of superintegrable systems developed mainly in ref. [4].

2. The ODE problem on supermanifolds

2.1. DEFINITIONS, CONVENTIONS, AND NOTATION

We shall refer the reader to refs. [3,5] for definitions. Our conventions are the following: A supermanifold shall always mean a real supermanifold; it is a pair, $(M, \mathcal{A}_M)$, with M some m-dimensional real smooth manifold, and $\mathcal{A}_M$ the structure sheaf of real superfunctions on M. A morphism $(M, \mathcal{A}_M) \to (N, \mathcal{A}_N)$ is a pair $\Psi = (\widetilde{\Psi}, \Psi^*)$, with $\widetilde{\Psi}\colon M \to N$ continuous, and $\Psi^*\colon \mathcal{A}_N(N) \to \mathcal{A}_M(\widetilde{\Psi}^{-1}(N))$, a map of $\mathbb{R}$-superalgebras commuting with restrictions. The terminal object is $(\{*\}, \mathbb{R})$; a point with the algebra of constants on it. The terminal morphism, $(M, \mathcal{A}_M) \to (\{*\}, \mathbb{R})$ shall be denoted by C. By definition, a supermanifold has a preferred embedding, $\delta\colon (M, \mathcal{C}^\infty_M) \to (M, \mathcal{A}_M)$; its superalgebra map, $\delta^*\colon \mathcal{A}_M(M) \to \mathcal{C}^\infty_M(M)$ shall be written, $f \mapsto \widetilde{f}$. Each point $x \in M$ defines a morphism $\delta_x\colon (\{*\}, \mathbb{R}) \to (M, \mathcal{A}_M)$, by letting, ${\delta_x}^*\colon \mathcal{A}_M(M) \to \mathbb{R}$, be, $f \mapsto \widetilde{f}(x)$. The composition $\delta_x \circ C$ gives a superalgebra map closely related to this: $f \mapsto \widetilde{f}(x) 1_{\mathcal{A}_M(M)}$. Note that the domain can be any supermanifold. We shall write C_x, and $\mathrm{ev}|_x$, instead of $\delta_x \circ C$, and $(\delta_x \circ C)^*$, respectively. In products, with underlying points $(x, y) \in M \times N$, the notation $\mathrm{ev}|_x$ stands for $(C_x \times \mathrm{id}_N)^*$; it pulls back superfunctions on $M \times N$ to superfunctions on N.

An *(m, n)-dimensional superdomain* shall always be understood as a *coordinate superdomain;* i.e., an open coordinate domain, U, in some $\mathbb{R}^m$, and the exterior bundle, $\bigwedge \mathbb{R}^n$, based on the trivial rank-n-bundle, $\mathbb{R}^n$, over U. When $U = \mathbb{R}^m$, the corresponding superdomain, $(\mathbb{R}^m, \Gamma(\bigwedge \mathbb{R}^n))$, shall be denoted by $\mathbb{R}^{m|n}$. It is convenient to write $\mathcal{U} \subset \mathbb{R}^{m|n}$ for the restriction of the structure sheaf $\Gamma(\bigwedge \mathbb{R}^n)$ on $\mathbb{R}^m$, to the open domain $U \subset \mathbb{R}^m$. Thus, $\mathcal{U} = (U, \Gamma(\bigwedge \mathbb{R}^n)|_U)$.

2.2. ODE'S ON SUPERDOMAINS

Let $(M, \mathcal{A}_M)$ be a supermanifold, and let $\mathrm{Der}\,\mathcal{A}_M$ be the sheaf of superderivations of $\mathcal{A}_M(M)$. Supervector fields on M are sections of $\mathrm{Der}\,\mathcal{A}_M$. Each supervector field X defines in a unique fashion a smooth vector field, $\widetilde{X} \in \mathrm{Der}\,\mathrm{C}^\infty(M)$, by letting, $\widetilde{X}(\widetilde{f}) = \widetilde{X_0(f)}$, for each $f \in \mathcal{A}_M(M)$ (cf. ref. [3, section 2.8]). It is well known that $\widetilde{X}$, gives rise to a collection of smooth maps, $\{\phi_t\}_{t\in\mathbb{R}}$, for which the following is true (cf. ref. [15]):

For each $t \in \mathbb{R}$, there exists a maximal open subset $V_t(\widetilde{X}) \subset M$, and a smooth map, $\phi_t\colon V_t(\widetilde{X}) \to M$, such that *(i)* $\phi_t(x) = \gamma(t,x)$, *for each* $x \in V_t(\widetilde{X})$; γ *being the unique integral curve of* $\widetilde{X}$ *through* x *at* $t = 0$, *and defined on its maximal domain* $\{t \in \mathbb{R} \mid x \in V_t(\widetilde{X})\}$.

(ii) ϕ_t: $V_t(\widetilde{X}) \to V_{-t}(\widetilde{X})$ *is a diffeomorphism whose inverse is* ϕ_{-t}.

(iii) $V_0(\widetilde{X}) = M$, $\phi_0 = id_M$, *and* $V_{t_1}(\widetilde{X}) \subset V_{t_2}(\widetilde{X})$, *if* $t_1 \geq t_2 \geq 0$, *or* $t_1 \leq t_2 \leq 0$.

(iv) $\bigcup_{t>0} V_t(\widetilde{X}) = \bigcup_{t<0} V_t(\widetilde{X}) = M$.

Furthermore, for each $x \in M$ *there is an open subset* $V_x(\widetilde{X}) \subset M$, *and some* $\epsilon > 0$, *such that the map* $(t,y) \mapsto \phi_t(y)$ *is smooth and defined on* $(-\epsilon,\epsilon) \times V_x(\widetilde{X})$.

In particular, the subset $V(\widetilde{X}) = \{(t,x) \in \mathbb{R} \times M \mid x \in V_t(\widetilde{X})\}$ is open, and a smooth map $\phi : V(\widetilde{X}) \to M$ can be defined by $\phi(t,x) = \phi_t(x)$; equivalently, there is a well defined homomorphism ϕ^*: $C^\infty(M) \to C^\infty(V(\widetilde{X}))$, $\phi^*(f) = f \circ \phi$, which is the unique solution to the equation

$$\widetilde{D} \circ \phi^* = \phi^* \circ \widetilde{X}, \tag{2.1}$$

subject to the initial condition $\phi(0,x) = x$. We have written $\widetilde{D}$ for the lift to $V(\widetilde{X})$ of the vector field $\mathrm{d}/\mathrm{d}t$ defined on $\mathbb{R}$. This lift is uniquely defined by the conditions $\widetilde{D} \circ p_1{}^* = p_1{}^* \circ \mathrm{d}/\mathrm{d}t$, and $\widetilde{D} \circ p_2{}^* = 0$; p_1, and p_2 being the projections of $V(\widetilde{X}) \subset \mathbb{R} \times M$ into the corresponding factors. Note that (2.1) is equivalent to the following equation in $C^\infty(M)$:

$$\mathrm{ev}|_{t=t_0} \circ \widetilde{D} \circ \phi^* = \mathrm{ev}|_{t=t_0} \circ \phi^* \circ \widetilde{X}, \tag{2.2}$$

for each $t_0 \in p_1(V(\widetilde{X}))$. Now let $\mathcal{V}_X$ be the open subsupermanifold of $\mathbb{R}^{1|1} \times (M, \mathcal{A}_M)$ whose underlying smooth manifold is $V(\widetilde{X})$. A solution to the $\mathbb{Z}_2$-graded differential equation defined by X, is a supermanifold morphism,

$$\Gamma\colon \mathcal{V}_X \to \mathcal{A}_M, \tag{2.3}$$

such that, for each $t_0 \in \mathbb{R}$,

$$\mathrm{ev}|_{t=t_0} \circ \widetilde{D} \circ \Gamma^* = \mathrm{ev}|_{t=t_0} \circ \Gamma^* \circ X, \tag{2.4}$$

subject to the initial condition,

$$\Gamma \circ (C_0 \times \mathrm{id}) = \mathrm{id}. \tag{2.5}$$

Equality (2.4) is set between superderivations of the sheaf $\mathcal{A}_M$. We have written $\widetilde{D}$ for the lift to $\mathcal{V}_X$ of a preferred superfield, D, on $\mathbb{R}^{1|1}$ (cf. section 2.4 below). This lift is defined by the conditions $\widetilde{D} \circ p_1{}^* = p_1{}^* \circ D$, and $\widetilde{D} \circ p_2{}^* = 0$; p_1, and p_2 being the projections of $\mathcal{V}_X$ into the corresponding factors (which are open subsupermanifolds $\mathbb{R}^{1|1}$ and $(M, \mathcal{A}_M)$, respectively). The evaluation morphism $\mathrm{ev}|_{t=t_0}$ is used to pull the superfunctions in $\mathcal{V}_X$ back to $(M, \mathcal{A}_M)$ (cf. refs. [5,9,1]). Note that the initial condition (2.5) may be rewritten as,

$$\mathrm{ev}|_{t=0}\, \Gamma^* f = f, \tag{2.6}$$

for any superfunction $f \in \mathcal{A}_M$.

2.3. ON THE ROLE OF THE EVALUATION MORPHISM

It is worth our while to actually appreciate the difference between eq. (2.4) and

$$\tilde{D} \circ \Gamma^* = \Gamma^* \circ X. \tag{2.7}$$

In order to do this we shall work on the supermanifold $\mathbb{R}^{m|n}$, and we shall write equations (2.7), and (2.4) in local coordinates for an arbitrary derivation D in $\mathbb{R}^{1|1}$. Let $\{t, \tau\}$ be a set of local coordinates in $\mathbb{R}^{1|1}$. There is no loss of generality in assuming that the integration model is locally of the form

$$D = (1 + \alpha\tau)\,\partial_t + (1 + \beta\tau)\,\partial_\tau, \tag{2.8}$$

where α, and β are smooth functions of t. Now, let $\{x^i; \theta^\mu\}$, be a set of local coordinates in $\mathcal{U} \subset \mathbb{R}^{m|n}$. Let X be some given supervector field on $\mathcal{U}$, and write it in these local coordinates as

$$\begin{aligned} X = &\sum_i \left(A^i + \sum_\mu A^i_\nu \theta^\nu + \sum_{\mu<\nu} A^i_{\mu\nu}\theta^\mu\theta^\nu + \cdots \right) \partial_{x^i} \\ &+ \sum_\rho \left(B^\rho + \sum_\nu B^\rho_\nu \theta^\nu + \sum_{\mu<\nu} B^\rho_{\mu\nu}\theta^\mu\theta^\nu + \cdots \right) \partial_{\theta^\rho}. \end{aligned} \tag{2.9}$$

Let Γ be a morphism $\mathbb{R}^{1|1} \times \mathcal{U} \to \mathcal{U}$. We shall write it in coordinates as

$$\begin{aligned} \Gamma^* x^i &= \gamma^i_0 + \sum_\nu \gamma^i_\nu\, \tau\,\theta^\nu + \sum_{\mu<\nu} \gamma^i_{\mu\nu}\,\theta^\mu\theta^\nu + \cdots\,,, \\ \Gamma^*\theta^\rho &= g^\rho_0\,\tau + \sum_\nu g^\rho_\nu\,\theta^\nu + \sum_{\mu<\nu} g^\rho_{\mu\nu}\,\tau\,\theta^\mu\theta^\nu + \sum_{\lambda<\mu<\nu} g^\rho_{\lambda\mu\nu}\,\theta^\lambda\theta^\mu\theta^\nu + \cdots\,, \end{aligned} \tag{2.10}$$

where, in fact, we should have written $p_1{}^*\tau$, and $p_2{}^*\theta^\mu$, instead of just τ, and θ, as we just did, for the local coordinates on $\mathbb{R}^{1|1} \times \mathcal{U}$ (p_j being the projection onto the jth-factor of the product $\mathbb{R}^{1|1} \times \mathcal{U}$). Let us further simplify the notation, and write

$$\begin{aligned} \Gamma^* x^i &= (\gamma^i_{(0)} + \gamma^i_{(2)} + \cdots) + \tau\,(\gamma^i_{(1)} + \gamma^i_{(3)} + \cdots), \\ \Gamma^*\theta^\mu &= \tau\,(g^\mu_{(0)} + g^\mu_{(2)} + \cdots) + (g^\mu_{(1)} + g^\mu_{(3)} + \cdots), \end{aligned} \tag{2.11}$$

where '(k)' denotes the $\mathbb{Z}$-degree of homogeneity in the odd variables $\{\theta^\mu\}$. Then,

$$\begin{aligned} \Gamma^* X x^i = &\left(\gamma_{(0)}{}^* A^i + \sum_\nu \gamma_{(0)}{}^* A^i_\nu g^\nu_{(1)} + \cdots\right) \\ &+ \tau\left(\sum_\nu \gamma_{(0)}{}^* A^i_\nu g^\nu_{(0)} + \sum_j \gamma_{(0)}{}^*\partial_{x^j} A^i \gamma^j_{(1)} + \cdots\right), \end{aligned}$$

$$\Gamma^* X\theta^\mu = (\gamma_{(0)}{}^* B^\mu + \sum_\nu \gamma_{(0)}{}^* B^\mu_\nu g^\nu_{(1)} + \cdots) + \tau\left(\sum_\nu \gamma_{(0)}{}^* B^\mu_\nu g^\nu_{(0)} + \sum_j \gamma_{(0)}{}^* \partial_{x^j} B^\mu \gamma^j_{(1)} + \cdots\right), \quad (2.12)$$

where use has been made of the fact that for any C^∞ function f on $\mathcal{U}$, $\Gamma^* f$ is given by $f \circ \gamma_{(0)} + \tau \sum \gamma^j_\nu \theta^\nu \partial_{x^j} f \circ \gamma_{(0)} + \cdots$. On the other hand, we have

$$\begin{aligned} \tilde{D}\Gamma^* x^i &= (\gamma^i_{(0)}{}' + \gamma^i_{(2)}{}' + \cdots) + (\gamma^i_{(1)} + \gamma^i_{(3)} + \cdots) \\ &\quad + \tau[\alpha(\gamma^i_{(0)}{}' + \gamma^i_{(2)}{}' + \cdots) + (\gamma^i_{(1)}{}' + \gamma^i_{(3)}{}' + \cdots) \\ &\quad + \beta(\gamma^i_{(1)} + \gamma^i_{(3)} + \cdots)] \\ \tilde{D}\Gamma^* \theta^\mu &= (g^\mu_{(0)} + g^\mu_{(2)} + \cdots) + (g^\mu_{(1)}{}' + g^\mu_{(3)}{}' + \cdots) \\ &\quad + \tau[(g^\mu_{(0)}{}' + g^\mu_{(2)}{}' + \cdots) + \alpha(g^\mu_{(1)}{}' + g^\mu_{(3)}{}' + \cdots) \\ &\quad + \beta(g^\mu_{(0)} + g^\mu_{(2)} + \cdots)]. \end{aligned} \quad (2.13)$$

If the definition of the superdifferential equation is given without the evaluation morphism in front of it, these local-coordinate expressions yield two separate systems of equations; namely

$$\begin{aligned} \gamma^i_{(0)}{}' &= \gamma_{(0)}{}^* A^i, & g^\mu_{(0)} &= \gamma_{(0)}{}^* B^\mu, \\ \gamma^i_{(1)} &= \sum_\nu \gamma_{(0)}{}^* A^i_\nu\, g^\nu_{(1)}, & g^\mu_{(1)}{}' &= \sum_\nu \gamma_{(0)}{}^* B^\mu_\nu\, g^\nu_{(1)}, \\ \cdots &= \cdots, & \cdots &= \cdots, \end{aligned} \quad (2.14)$$

and

$$\begin{aligned} \alpha\gamma^i_{(0)}{}' &= \sum_\nu \gamma_{(0)}{}^* A^i_\nu\, g^\nu_{(0)}, & g^\mu_{(0)}{}' + g^\mu_{(0)} &= \sum_\nu \gamma_{(0)}{}^* B^\mu_\nu\, g^\nu_{(0)}, \\ \gamma^i_{(1)}{}' + \beta\gamma^i_{(1)} &= \sum_j \gamma_{(0)}{}^* \partial_{x^j} A^i \gamma^j_{(1)}, & \alpha g^\mu_{(1)}{}' &= \sum_j \gamma_{(0)}{}^* \partial_{x^j} B^\mu \gamma^j_{(1)}, \\ \cdots &= \cdots, & \cdots &= \cdots. \end{aligned} \quad (2.15)$$

It is intuitively clear from these expressions that a unique solution exists to the first set of equations: The first equation in (2.14) is classical. Its unique solution, $\gamma_{(0)}$, may be plugged into the equation right in front of it to determine uniquely the coefficients $g^\mu_{(0)}$. In fact, $\gamma_{(0)}$ must be plugged into every single equation of the system. The next equation to solve is the second on the right in order to determine uniquely the $g^\mu_{(1)}$'s. Then go to the equation on the left to determine the $\gamma^i_{(1)}$'s in terms of the $g^\mu_{(1)}$'s. It is clear that this "shoe-lace" manner of solving the first set completely determines the coefficients of the flow in a unique fashion. Now, the second set of equations arises from the coefficients of τ. Since the flow coefficients are already determined, the second set must be thought of as identities that ought to be true among the coefficients of the given superfield. If no restrictions are imposed on the superfields to be integrated, the second set of equations should not be there at all. The way to do this is precisely by formulating the differential equation of the superfield X with the evaluation morphism in it, so that the equality (2.7) really means a congruence $(\mathrm{mod}\,\tau)$; but this is precisely what eq. (2.4) says. We shall see in theorem 3.6 below precisely under what circumstances the superdifferential equation can be posed without the ev-map.

2.4. ON THE CHOICE OF THE INTEGRATING MODEL

Note that once the ev-map is made part of the definition, the system of equations obtained from the model D, and that obtained from the model $D + \tau D'$ are exactly the same. This can be readily seen, either from the fact that $\mathrm{ev}|_{t=t_0}\, \tau = 0$, or from the coordinate expressions above. (Note that the system (2.14) obtained from the full derivation D, is exactly the same as the one obtained from $\partial_t + \partial_\tau$).

Now, one may argue that if some pair of homogeneous fields, D_0, and D_1, is chosen as model for the integration of all supervector fields, they must form a Lie superalgebra. In fact, D_0, and D_1 must generate the Lie superalgebra of left invariant supervector fields on a $(1,1)$-dimensional Lie supergroup. If furthermore, the integration theory of supervector fields is required to reproduce the C^∞ theory under the canonical morphism $\mathcal{A}_M \to \mathcal{C}^\infty_M$, the underlying Lie group must be $\mathbb{R}$ with its additive structure. It is well known (and easy to see) that up to isomorphism there are only three $(1,1)$-dimensional Lie superalgebras; labeling them with $\mathrm{j} = 1$, 2, and 3, their structure may be displayed as follows (multiplying D_0 by a constant if necessary, one may assume that $a = 1$, or $b = 1$ in (1.4)):

$$[D_0, D_1] = \delta_{\mathrm{j}3}\, D_1 \quad \text{and} \quad [D_1, D_1] = \delta_{\mathrm{j}2}\, D_0$$

To realize D_0, and D_1 as supervector fields satisfying these commutation relations on the $(1,1)$-dimensional Lie supergroup, local coordinates $\{t, \tau\}$ may be chosen in such a way that

$$D_0 = \partial_t + a\tau\partial_\tau \quad \text{and} \quad D_1 = \partial_\tau + b\tau\partial_t, \tag{2.16}$$

where a and b are real constants satisfying $a\,b = 0$. (If the constraint on D_0, and D_1, to generate one of the Lie superalgebras above is not imposed, a, and b would be arbitrary smooth functions of t). By formal exponentiation of the Lie superalgebra elements one obtains formally some 'Lie supergroup elements' from which the supergroup operation may be obtained. Now, for the sake of clarity we shall state first the explicit operations, μ_{j}: $\mathbb{R}^{1|1} \times \mathbb{R}^{1|1} \to \mathbb{R}^{1|1}$ ($\mathrm{j} = 1$, 2, and 3) that endow $\mathbb{R}^{1|1}$ with a Lie supergroup structure (propostion 2.1 below), and recover a posteriori the corresponding left invariant supervector fields (section

5 below). Thus, we have the following result whose proof is a straightforward verification using the techniques of ref. [1]:

Proposition 2.1. *Let* $\{t,\tau\}$ *be the standard supercoordinate system on* $\mathbb{R}^{1|1}$ *(i.e., t is the linear functional* $\mathbb{R}\to\mathbb{R}$ *dual to the basis element* $1\in\mathbb{R}$*, and* τ *is the generator of* $\bigwedge(\mathbb{R}t)^*$ *dual to* t^**). There are three different supergroup structures on* $\mathbb{R}^{1|1}$ *whose composition morphisms,* μ_j: $\mathbb{R}^{1|1}\times\mathbb{R}^{1|1}\to\mathbb{R}^{1|1}$ *(*$j=1,2,3$*), are respectively given by the following superalgebra maps:*

$$(\mu_j{}^*t,\mu_j{}^*\tau)=\begin{cases}(p_1{}^*t+p_2{}^*t,p_1{}^*\tau+p_2{}^*\tau); & j=1,\\ (p_1{}^*t+p_2{}^*t+p_1{}^*\tau\, p_2{}^*\tau,p_1{}^*\tau+p_2{}^*\tau); & j=2,\\ (p_1{}^*t+p_2{}^*t,e^{p_2{}^*t}p_1{}^*\tau+p_2{}^*\tau); & j=3,\end{cases}\tag{2.17}$$

where p_1*, and* p_2 *are the projection morphisms of* $\mathbb{R}^{1|1}\times\mathbb{R}^{1|1}$ *into their factors. In all cases, the identity morphism is given by* $ev|_{t=0}$*, and the inversion superdiffeomorphism* α: $\mathbb{R}^{1|1}\to\mathbb{R}^{1|1}$ *is given by*

$$(\alpha_j{}^*t,\alpha_j{}^*\tau)=\begin{cases}(-t,-\tau); & j=1,2,\\ (-t,-e^{-t}\tau); & j=3,\end{cases}\tag{2.18}$$

Remark. It is shown in section 5 below that the homogeneous generators for the corresponding Lie superalgebras of left invariant supervector fields are given by

$$D_0=\begin{cases}\partial_t & \mathrm{j}=1,2,\\ \partial_t+\tau\partial_\tau & \mathrm{j}=3,\end{cases}\quad\text{and}\quad D_1=\begin{cases}\partial_\tau & \mathrm{j}=1,3,\\ \partial_\tau+\tau\partial_t & \mathrm{j}=2.\end{cases}\tag{2.19}$$

In what follows, it will be assumed that $D=\partial_t+\partial_\tau+\tau D_j{}'$, with,

$$D_{\mathrm{j}}{}'=\delta_{\mathrm{j}2}\partial_t+\delta_{\mathrm{j}3}\partial_\tau;\quad \mathrm{j}=1,2,3.\tag{2.20}$$

3. Existence and uniqueness of solutions to super-ODE's

By Batchelor's theorem, any supermanifold $(M,\mathcal{A}_M)$ is isomorphic (although not canonically) to a supermanifold of the form $(M,\Gamma(\bigwedge E))$, where π: $E\to M$ is a smooth vector bundle. The proof of existence and uniqueness of solutions to super-ODE's in a supermanifold $(M,\mathcal{A}_M)$ can be reduced to the same problem in a graded manifold of the Batchelor kind. This is a consequence of the following.

Lemma 3.1. *Let* $\sigma:(M,\mathcal{A}_M)\to(N,\mathcal{A}_N)$ *be a supermanifold isomorphism. Let* X *be a supervector field on* $(M,\mathcal{A}_M)$ *and let* $\mathcal{V}_X$ *be the maximal domain of* *definition of some solution, Γ: $\mathcal{V}_X \to (M, \mathcal{A}_M)$, to the ODE defined by X*

$$ev|_{t=t_0} \circ \widetilde{D} \circ \Gamma^* = ev|_{t=t_0} \circ \Gamma^* \circ X. \tag{3.1}$$

Let $\sigma_* X = (\sigma^{-1})^* \circ X \circ \sigma^*$ be the supervector field on $(N, \mathcal{A}_N)$ induced by σ and X, and let p_1, and p_2 be the projections of the product $\mathbb{R}^{1|1} \times \mathcal{M}$ onto their corresponding factors. Then, $\sigma \circ \Gamma \circ (p_1 \times \sigma^{-1} \circ p_2)$ is a solution to the ODE defined by $\sigma_* X$, and its domain of definition, $\sigma(\mathcal{V}_X)$, is maximal. Furthermore, if Γ is a unique solution satisfying (2.6), then $\sigma \circ \Gamma \circ (p_1 \times \sigma^{-1} \circ p_2)$ is also a unique solution satisfying a similar condition for superfunctions in $\mathcal{A}_N$.

Proof. We shall write $\widetilde{D}^M$, and $\widetilde{D}^N$, for the lifts of D to $\mathbb{R}^{1|1} \times (M, \mathcal{A}_M)$, and $\mathbb{R}^{1|1} \times (N, \mathcal{A}_N)$, respectively. The fact that $\sigma \circ \Gamma \circ (p_1 \times \sigma^{-1} \circ p_2)$ is a solution to the equation defined by $\sigma_* X$, is a consequence of the following equalities:

$$\begin{aligned} &\mathrm{ev}|_{t=t_0} \circ \widetilde{D}^N \circ (p_1 \times \sigma^{-1} \circ p_2)^* \circ \Gamma^* \circ \sigma^* \\ &\quad = \mathrm{ev}|_{t=t_0} \circ (p_1 \times \sigma^{-1} \circ p_2)^* \circ \widetilde{D}^M \circ \Gamma^* \circ \sigma^* \\ &\quad = (\sigma^{-1})^* \circ \mathrm{ev}|_{t=t_0} \circ \widetilde{D}^M \circ \Gamma^* \circ \sigma^* \\ &\quad = (\sigma^{-1})^* \circ \mathrm{ev}|_{t=t_0} \circ \Gamma^* \circ X \circ \sigma^* \\ &\quad = \mathrm{ev}|_{t=t_0} \circ (p_1 \times \sigma^{-1} \circ p_2)^* \circ \Gamma^* \circ \sigma^* \circ (\sigma^{-1})^* \circ X \circ \sigma^* \\ &\quad = \mathrm{ev}|_{t=t_0} \circ (p_1 \times \sigma^{-1} \circ p_2)^* \circ \Gamma^* \circ \sigma^* \circ (\sigma_* X), \end{aligned}$$

where the following facts have been used:

$$\widetilde{D}^N \circ (p_1 \times \sigma^{-1} \circ p_2)^* = (p_1 \times \sigma^{-1} \circ p_2)^* \circ \widetilde{D}^M,$$

$$\mathrm{ev}|_{t=t_0} \circ (p_1 \times \sigma^{-1} \circ p_2)^* = (\sigma^{-1})^* \circ \mathrm{ev}|_{t=t_0}.$$

The maximality of the domain can be easily deduced arguing by contradiction, and using the facts that σ is an isomorphism, and that $\mathcal{V}_X$ was assumed to be maximal. The uniqueness part of the statement is proved similarly. □

Remark. This lemma implies that if we know how to integrate ODE's in Batchelor supermanifolds, then we also know how to integrate them in any supermanifold. Moreover, the non-canonicity of the Batchelor isomorphism is not a problem: Let us suppose that $\sigma_i : (M, \mathcal{A}_M) \to (M, \Gamma(\bigwedge E_i))$, $i = 1, 2$, are two isomorphisms and that X is a supervector field on $(M, \mathcal{A}_M)$. Then, the supervector fields $(\sigma_1)_* X$ and $(\sigma_2)_* X$ are related by the isomorphism $\sigma_2 \circ (\sigma_1)^{-1}$; hence, by the lemma, their integrals are also related, and once the uniqueness question is settled (cf. prop. 3.2, and theorem 3.5 below), they will define the same solution in $(M, \mathcal{A}_M)$.

From now on we shall assume that $(M, \mathcal{A}_M) = (M, \Gamma(\bigwedge E))$, where $\pi : E \to M$ is a smooth vector bundle. For the sake of simplicity, we shall occasionally write $\mathcal{M}$ for the pair $(M, \mathcal{A}_M)$. Let $X \in \operatorname{Der} \mathcal{A}_M(M)$ be a supervector field, and assume Γ is a morphism, $\mathcal{V}_X \to \mathcal{M}$, satisfying (2.4), and (2.5). As we pointed it out in the introduction, the fact that Γ^*, and $\mathrm{ev}|_{t=t_0}$ are morphisms of superalgebras, makes it possible to split the differential equation (2.4) into two equations:

$$\begin{aligned} \mathrm{ev}|_{t=t_0} \circ \widetilde{D_0} \circ \Gamma^* &= \mathrm{ev}|_{t=t_0} \circ \Gamma^* \circ X_0 \quad \text{and} \\ \mathrm{ev}|_{t=t_0} \circ \widetilde{D_1} \circ \Gamma^* &= \mathrm{ev}|_{t=t_0} \circ \Gamma^* \circ X_1, \end{aligned} \tag{3.2}$$

where $X = X_0 + X_1$, and $D = D_0 + D_1$, are the corresponding $\mathbb{Z}_2$-decompositions of X, and D. We shall assume that D_0, and D_1 are the generators of the left-invariant supervector fields on $\mathbb{R}^{1|1}$ for one of the Lie supergroup structures listed in proposition 2.1. Due to the first observation made in section 2.4, the integral flow Γ only depends on the congruence class of D_0, and D_1, modulo τ; whence only on ∂_t, and ∂_τ, respectively.

Now, for the proof of the existence and uniqueness theorem we shall follow the methods of ref. [7]. Let $\overline{X}$ be the supervector field on $\mathcal{V}_X$ defined by the conditions $\overline{X} \circ p_1{}^* = 0$ and $\overline{X} \circ p_2{}^* = p_2{}^* \circ X$. Since $p_2{}^*$ is monic, it is easy to see that the map $X \mapsto \overline{X}$ is monic. We shall make use of the Type 1 structure in our first few results, but only to apply the main result of ref. [7, proposition 3.2], and to find explicitly the integral flow in terms of an auxiliary Type 1-$\mathbb{R}^{1|1}$-action, Φ (see theorem 3.5 below). Once this is done, the question of whether or not the integral flow defines a Lie supergroup action can be stated as in theorem 3.6 below, without having to compromise with any specific choice.

Proposition 3.2. *Let X_0 be an even supervector field on $(M, \mathcal{A}_M) = (M, \Gamma(\bigwedge E))$, and let the notation be as in section 2.3 above. There exists a unique solution Φ: $\mathcal{V}_{X_0} \to \mathcal{M}$, to the equation*

$$\widetilde{\partial_t + \partial_\tau} \circ \Phi^* = \Phi^* \circ X_0,$$

satisfying the initial condition $ev|_{t=0} \circ \Phi^* = \mathrm{id}$. Furthermore, the solution Φ satisfies the following properties:

(i) $\Phi^* \circ X_0 = \overline{X_0} \circ \Phi^*$.

(ii) Φ defines a Type 1-$\mathbb{R}^{1|1}$-action.

Proof. This is simply theorem 3 of ref. [7]. The only subtle point is this: According to ref. [7], Φ defines a (local) $\mathbb{R}$-action. The statement that it defines in fact a Type 1-$\mathbb{R}^{1|1}$-action follows easily because $\widetilde{\partial_\tau} \circ \Phi^* = 0$. □

Remark. The proof of this result in ref. [7] was carried out with the help of a linear connection defined on the bundle E. The connection was only used to have manageable expressions of supervector fields as derivations. Its role is unessential; in fact, the solution found turns out to be independent of the linear connection, as expected (e.g., one may argue by uniqueness). On the other hand, it is interesting to note that an essential use of the underlying C^∞ flow of $\widetilde{X}$ is made in ref. [7] in order to determine the maximal domain of the solution. Now, if one proceeds naively integrating the local expressions (2.14) one can easily get convinced that, up to first order in the odd variables, there will be a unique solution Γ of (3.2), such that $\widetilde{\Gamma} = \gamma_{(0)}$ is the smooth flow on the coordinate neighborhood U generated by $\widetilde{X} = \sum A^i \partial_{x^i}$, and $g_{(1)}$ is the parallel transport on E with respect to a connection ∇ uniquely determined by the order-one coefficients of the odd part of the field X. In fact, one may think of $t \mapsto g_{(1)}(t)$ as a curve on $\operatorname{End} E$, which in view of the initial condition (2.6) goes through the identity at $t = 0$. But then, the differential equation for $g_{(1)}$ in (2.14) is simply the equation that defines parallel transport on E along $\gamma_{(0)}$, with respect to the connection form whose matrix is (B^μ_ν).

We shall now turn to the integration of odd supervector fields.

Lemma 3.3. *Let X_1 be an odd supervector field in $(M, \mathcal{A}_M)$. Let $X_0 \in \operatorname{Der} \mathcal{A}_M(M)$ be even, and let $\Phi: \mathcal{V} \to \mathcal{M} = (M, \mathcal{A}_M)$ be its unique integral flow as in proposition 3.2 ($\mathcal{V} \subset \mathbb{R}^{1|1} \times \mathcal{M}$). Let p_1, and p_2 be the projections of the product $\mathbb{R}^{1|1} \times \mathcal{M}$ onto their corresponding factors, and let $\alpha: \mathbb{R}^{1|1} \to \mathbb{R}^{1|1}$ be the inversion superdiffeomorphism on the Type 1 Lie supergroup structure of $\mathbb{R}^{1|1}$ (cf. (2.20) above). Define,*

$$\Phi_\alpha = (p_1 \times \Phi) \circ (\alpha \circ p_1 \times p_2) : \mathcal{V} \to \mathcal{V}.$$

Then,

(i) Φ_α is a superdiffeomorphism whose inverse is ${\Phi_\alpha}^{-1} = p_1 \times \Phi$.

(ii) $\widetilde{\partial}_t \circ {\Phi_\alpha}^{-1^*} \circ \overline{X}_1 \circ {\Phi_\alpha}^* = {\Phi_\alpha}^{-1^*} \circ [\overline{X_0, X_1}] \circ {\Phi_\alpha}^*$.

(iii) The integral flow of ${\Phi_\alpha}^{-1^*} \circ \overline{X}_1 \circ {\Phi_\alpha}^*$ is $\rho : \mathcal{W} \to \mathcal{V}$, with

$$\rho^* = {\pi_2}^* + ({\pi_1}^* \tau)\,({\pi_2}^* \circ {\Phi_\alpha}^{-1^*} \circ \overline{X}_1 \circ {\Phi_\alpha}^*);$$

$\mathcal{W} \subset \mathbb{R}^{1|1} \times \mathcal{V}$, and π_1, and π_2 the projections of $\mathbb{R}^{1|1} \times \mathcal{V}$ into their corresponding factors.

Proof. The first assertion follows from the fact that Φ is an action: Indeed (cf. ref. [1]),

$$\begin{aligned}(p_1 \times \Phi \circ (\alpha \circ p_1 \times p_2)) \circ (p_1 \times \Phi) &= p_1 \times \Phi \circ (\alpha \circ p_1 \times \Phi) \\ &= p_1 \times \Phi \circ (C_0 \times p_2) = p_1 \times p_2.\end{aligned}$$

The second assertion requires a little work to establish the following facts:

(a) $\widetilde{\partial}_t \circ (p_1 \times \Phi)^* = (p_1 \times \Phi)^* (\widetilde{\partial}_t + \overline{X}_0)$, i.e., $(p_1 \times \Phi)$ is the integral flow of $\widetilde{\partial}_t + \overline{X}_0$, with no need of the ev-morphism (cf. proposition 3.2 above).

(b) $\widetilde{\partial}_t \circ \overline{X}_1 = \overline{X}_1 \circ \widetilde{\partial}_t$.
(c) $(\alpha \circ p_1 \times p_2)^* \circ \overline{X}_0 = \overline{X}_0 \circ (\alpha \circ p_1 \times p_2)^*$
(d) $(\alpha \circ p_1 \times p_2)^* \circ \widetilde{\partial}_t = -\widetilde{\partial}_t \circ (\alpha \circ p_1 \times p_2)^*$.

Now, these may be proved by showing that both sides of the stated equalities yield the same answer when applied to an arbitrary element of the form $p_1{}^*f\, p_2{}^*g$, with f a superfunction in $\mathbb{R}^{1|1}$, and g a superfunction in (M, A_M). Thus, for (a), we have

$$\begin{aligned}\widetilde{\partial}_t \circ (p_1 \times \Phi)^*(p_1{}^*f\, p_2{}^*g) &= \widetilde{\partial}_t(p_1{}^*f\, \Phi^*g)\\ &= (p_1{}^*(\partial_t f))\,(\Phi^*g) + (p_1{}^*f)\,(\widetilde{\partial}_t \circ \Phi^*g)\\ &= (p_1{}^*(\partial_t f))\,(\Phi^*g) + (p_1{}^*f)\,(\Phi^* \circ X_0\, g).\end{aligned}$$

On the other hand,

$$\begin{aligned}(p_1 \times \Phi)^* \circ (\widetilde{\partial}_t + \overline{X}_0)(p_1{}^*f\, p_2{}^*g) &= (p_1 \times \Phi)^*(p_1{}^*(\partial_t f))p_2{}^*g\\ &\quad + (p_1 \times \Phi)^* p_1{}^*f\,(p_2{}^*X_0\, g)\\ &= (p_1{}^*(\partial_t f))(\Phi^*g) + (p_1{}^*f)(\Phi^* \circ X_0 g).\end{aligned}$$

Similarly, for (b):

$$\widetilde{\partial}_t \circ X_1(p_1{}^*f\, p_2{}^*g) = [p_1{}^*\partial_t(f_0 - f_1\tau)]\, p_2{}^*X_1\, g = X_1 \circ \widetilde{\partial}_t(p_1{}^*f\, p_2{}^*g),$$

where we have written $f = f_0 + f_1\tau$, with f_0, anf f_1 smooth functions on $\mathbb{R}$. The proof of (c) is equally easy, and the proof of (d) comes down to show that $\partial_t\alpha^*f = -\alpha^*\partial_t f$, for any superfunction in $\mathbb{R}^{1|1}$. But this follows from the fact that $\alpha^*t = -t$, and $\alpha^*\tau = -\tau$. Indeed,

$$\begin{aligned}\partial_t\alpha^*f &= \partial_t(f_0 \circ \widetilde{\alpha} - f_1 \circ \widetilde{\alpha}\,\tau)\\ &= -f_0' \circ \widetilde{\alpha} + f_1' \circ \widetilde{\alpha}\,\tau\\ &= -\alpha^*(f_0' + f_1'\,\tau).\end{aligned}$$

With (a)–(d) settled, the second statement is a straightforward computation:

$$\begin{aligned}\widetilde{\partial}_t \circ {\Phi_\alpha}^{-1^*} \circ \overline{X}_1 \circ {\Phi_\alpha}^* &= \widetilde{\partial}_t \circ (p_1 \times \Phi)^* \circ \overline{X}_1 \circ {\Phi_\alpha}^*\\ &= (p_1 \times \Phi)^* \circ \widetilde{\partial}_t \circ \overline{X}_1 \circ {\Phi_\alpha}^* + (p_1 \times \Phi)^* \circ \overline{X}_0 \circ \overline{X}_1 \circ {\Phi_\alpha}^*\\ &= {\Phi_\alpha}^{-1^*} \circ \overline{X}_0 \circ \overline{X}_1 \circ {\Phi_\alpha}^* + {\Phi_\alpha}^{-1^*} \circ \overline{X}_1 \circ \widetilde{\partial}_t \circ (\alpha \circ p_1 \times p_2)^* \circ \Phi^*\\ &= {\Phi_\alpha}^{-1^*} \circ \overline{X}_0 \circ \overline{X}_1 \circ {\Phi_\alpha}^* - {\Phi_\alpha}^{-1^*} \circ \overline{X}_1 \circ (\alpha \circ p_1 \times p_2)^* \circ \widetilde{\partial}_t \circ \Phi^*\\ &= {\Phi_\alpha}^{-1^*} \circ \overline{X}_0 \circ \overline{X}_1 \circ {\Phi_\alpha}^* - {\Phi_\alpha}^{-1^*} \circ \overline{X}_1 \circ (\alpha \circ p_1 \times p_2)^* \circ \Phi^* \circ X_0\\ &= {\Phi_\alpha}^{-1^*} \circ \overline{X}_0 \circ \overline{X}_1 \circ {\Phi_\alpha}^* - {\Phi_\alpha}^{-1^*} \circ \overline{X}_1 \circ (\alpha \circ p_1 \times p_2)^* \circ \overline{X}_0 \circ \Phi^*\\ &= {\Phi_\alpha}^{-1^*} \circ \left(\overline{X}_0 \circ \overline{X}_1 - \overline{X}_1 \circ \overline{X}_0\right) \circ {\Phi_\alpha}^*.\end{aligned}$$

Finally, the third assertion is just a straightforward check: On the one hand,

$$(\widetilde{\partial_t + \partial_\tau}) \circ \rho^* = \pi_2{}^*\Phi_\alpha{}^{-1^*}\overline{X}_1\Phi_\alpha{}^* + (\pi_1{}^*\tau)\left(\pi_2{}^*\Phi_\alpha{}^{-1^*}[\overline{X_0, X_1}]\,\Phi_\alpha{}^*\right).$$

On the other hand,

$$\rho^* \circ \Phi_\alpha{}^{-1^*}\overline{X}_1\ \Phi_\alpha{}^* = \pi_2{}^*\Phi_\alpha{}^{-1^*}\overline{X}_1\ \Phi_\alpha{}^* + \frac{1}{2}(\pi_1{}^*\tau)\left(\Phi_\alpha{}^{-1^*}[\overline{X_1, X_1}]\,\Phi_\alpha{}^*\right),$$

and it is clear from both equations that the right hand sides are congruent (mod $\pi_1{}^*\tau$). □

Corollary 3.4. *Let the hypotheses be as in the previous lemma. The necessary and sufficient conditions for having*

$$(\widetilde{\partial_t + \partial_\tau}) \circ \rho^* = \rho^* \circ (\Phi_\alpha{}^{-1^*}\overline{X}_1\ \Phi_\alpha{}^*),$$

without the ev-morphism acting from the left are

$$[X_0, X_1] = 0 \quad \textit{and} \quad [X_1, X_1] = 0.$$

Proof. The right hand sides of the two equations in the proof of the previous lemma are equal if and only if

$$\Phi_\alpha{}^{-1^*} \circ [\overline{X_0, X_1}] \circ \Phi_\alpha{}^* = \tfrac{1}{2}\ \Phi_\alpha{}^{-1^*} \circ [\overline{X_1, X_1}] \circ \Phi_\alpha{}^*.$$

Since both sides of this equation are homogeneous elements of different parity, since $\Phi_\alpha{}^*$, and $\Phi_\alpha{}^{-1^*}$ are isomorphisms, and since $X \mapsto \overline{X}$ is monic, the assertion follows. □

Theorem 3.5. *Let X be a supervector field on (M, A_M), and let X_0, and X_1 be its homogeneous components. Let $\Phi\colon \mathcal{V} \to \mathcal{M} = (M, A_M)$ be the unique integral flow of X_0 as in proposition 3.2 ($\mathcal{V} \subset \mathbb{R}^{1|1} \times \mathcal{M}$). Let Φ_α be as in lemma 3.3, and let $\rho\colon \mathcal{W} \to \mathcal{V}$ ($\mathcal{W} \subset \mathbb{R}^{1|1} \times \mathcal{V}$) be the unique integral flow of $\Phi_\alpha{}^{-1^*} \circ \overline{X}_1 \circ \Phi_\alpha{}^*$. Then, there is a unique integral flow, Γ, of X satisfying the initial condition $ev|_{t=0} \circ \Gamma^* = \mathrm{id}$. In fact,*

$$\Gamma^* = \eta^* \circ \rho^* \circ \Phi^*,$$

where $\eta\colon \mathcal{V} \to \mathcal{W}$ is the unique morphism defined by the conditions

$$\eta^*\pi_1{}^*f = p_1{}^*f \quad \textit{and} \quad \eta^*\pi_2{}^*g = g,$$

for all superfunctions, f in $\mathbb{R}^{1|1}$, and g in $\mathcal{V}$.

Proof. Note that

$$\Phi \circ \Phi_\alpha = \Phi \circ \left(p_1 \times \Phi \circ (\alpha \circ p_1 \times p_2)\right) = \Phi \circ (C_0 \times p_2) = p_2,$$

because Φ is an $\mathbb{R}^{1|1}$-action. Therefore, ${\Phi_\alpha}^* \Phi^* = {p_2}^*$, and hence

$$\begin{aligned} \eta^* \rho^* \Phi^* &= \eta^*\Big({\pi_2}^*\Phi^* + ({\pi_1}^*\tau)\,({\pi_2}^*{\Phi_\alpha}^{-1^*}\overline{X}_1 {p_2}^*)\Big) \\ &= \eta^*\Big({\pi_2}^*\Phi^* + ({\pi_1}^*\tau)\,({\pi_2}^*{\Phi_\alpha}^{-1^*} {p_2}^* X_1)\Big) \\ &= \eta^*\Big({\pi_2}^*\Phi^* + ({\pi_1}^*\tau)\,({\pi_2}^*\Phi^* X_1)\Big) \\ &= \Phi^* + ({p_1}^*\tau)\,\Phi^* X_1, \end{aligned}$$

where use has been made of ${\Phi_\alpha}^{-1^*}{p_2}^* = (p_2 \circ {\Phi_\alpha}^{-1})^* = \Phi^*$ (cf. lemma 3.3-(1)). Now, on the one hand we have

$$(\widetilde{\partial_t + \partial_\tau}) \circ \Gamma^* = \Phi^* \circ (X_0 + X_1) + ({p_1}^*\tau)\,(\Phi^* \circ X_0 \circ X_1);$$

whereas,

$$\Gamma^* \circ (X_0 + X_1) = \Phi^* \circ (X_0 + X_1) + ({p_1}^*\tau)\,(\Phi^* \circ X_1 \circ (X_0 + X_1)).$$

It is now clear that the right hand sides of both equations are congruent (mod ${p_1}^*\tau$). □

Theorem 3.6. *Let X be a supervector field on $(M, \mathcal{A}_M)$, and let X_0, and X_1 be its homogeneous components. Let $j = 1$, 2, and, 3, label the different Lie supergroup structures of $\mathbb{R}^{1|1}$ as in proposition 2.1. Then, the following assertions are equivalent:*

(i) X_0, and X_1 generate the following $(1,1)$-dimensional Lie superalgebra:

$$[X_0, X_1] = \delta_{j3}\, X_1 \quad \textit{and} \quad [X_1, X_1] = \delta_{j2}\, X_0 \quad (j = 1, 2, 3).$$

(ii) The integral flow Γ of X satisfies the equation

$$(\partial_t + \widetilde{\partial_\tau} + D_j') \circ \Gamma^* = \Gamma^* \circ X,$$

without the ev-morphism (D_j' as in (2.20)).

(iii) The integral flow Γ of X defines a Type j-$\mathbb{R}^{1|1}$-action on $(M, \mathcal{A}_M)$.

The next-to-last equation in the proof of theorem 3.5 above implies that

$$\widetilde{\tau\partial_t} = ({p_1}^*\tau)\,(\Phi^* \circ X_0) \quad \text{and} \quad \widetilde{\tau\partial_\tau} = ({p_1}^*\tau)\,(\Phi^* \circ X_1).$$

It then follows that $(\partial_t + \widetilde{\partial_\tau} + D_j') \circ \Gamma^*$ is actually equal to (and not just congruent (mod ${p_1}^*\tau$) to) $\Gamma^* \circ (X_0 + X_1)$, if and only if

$$\begin{aligned} \Phi^* \circ X_0 \circ X_1 &= \Phi^* \circ X_1 \circ X_0 - \delta_{j,3}\Phi^* \circ X_1 \quad \text{and} \\ \Phi^* \circ X_1 \circ X_1 &= \delta_{j,2}\Phi^* \circ X_0, \end{aligned}$$

and the equivalence between (i), and (ii) follows.

To prove the equivalence between (i) and (iii) we shall need the specific Lie supergroup structures of $\mathbb{R}^{1|1}$ (cf. proposition 2.1 above). Let $\{t_1, t_2, \tau_1, \tau_2\}$ be the graded coordinates of $\mathbb{R}^{1|1} \times \mathbb{R}^{1|1}$ and $\{t, \tau\}$ those of $\mathbb{R}^{1|1}$. The map μ_j can be conveniently expressed in terms of μ_1 as follows:

$$\mu_j{}^* f = \mu_1{}^* f + \delta_{j2}\, \tau_1\, \tau_2\, \mu_1{}^* \circ (\partial_t) f + \delta_{j3}\, \tau_1\, (e^{t_2} - 1)\, \mu_1{}^* \circ (\partial_\tau) f,$$

for all $f \in \mathcal{A}_{\mathbb{R}^{1|1}}(\mathbb{R})$.

Now, let the notation be as in theorem 3.5 above. The integral flow Γ of X defines a Type j $\mathbb{R}^{1|1}$-action on $\mathcal{M} = (M, \mathcal{A}_M)$ iff the following diagram commutes:

$$\begin{array}{ccc} \mathcal{W} & \xrightarrow{\pi_1 \times \Gamma \circ \pi_2} & \mathcal{V} \\ {\scriptstyle \mu_j \circ (\pi_1 \times p_1 \circ \pi_2) \times p_2 \circ \pi_2} \downarrow & & \downarrow {\scriptstyle \Gamma} \\ \mathcal{V} & \xrightarrow{\Gamma} & \mathcal{M} \end{array} \quad . \tag{3.3}$$

We shall then need the following formula:

$$\begin{aligned} [\mu_j \circ (\pi_1 \times p_1 \circ \pi_2) \times p_2 \circ \pi_2]^* &= [\mu_1 \circ (\pi_1 \times p_1 \circ \pi_2) \times p_2 \circ \pi_2]^* \\ &+ \delta_{j,2}\, \tau_1\, \tau_2\, [\mu_1 \circ (\pi_1 \times p_1 \circ \pi_2) \times p_2 \circ \pi_2]^* \circ \widetilde{\partial}_t \\ &+ \delta_{j3}\, (e^{t_2} - 1)\, \tau_1\, [\mu_1 \circ (\pi_1 \times p_1 \circ \pi_2) \times p_2 \circ \pi_2]^* \circ \widetilde{\partial}_\tau, \end{aligned}$$

where we have written $\pi_1{}^*\tau = \tau_1$, $\pi_2{}^* p_1{}^* t = t_2$, and, $\pi_2{}^* p_1{}^* \tau = \tau_2$. This may be proved in a straightforward manner by the methods of lemma 3.3.

Since Φ is a Type 1 action, the diagram above commutes when Φ is placed instead of Γ, and $\mu_j = \mu_1$. Therefore, on the one hand we obtain

$$\begin{aligned} &[\mu_j \circ (\pi_1 \times p_1 \circ \pi_2) \times p_2 \circ \pi_2]^* \Gamma^* \\ &= [\mu_j \circ (\pi_1 \times p_1 \circ \pi_2) \times p_2 \circ \pi_2]^* (\Phi^* + (p_1{}^*\tau)(\Phi^* X_1)) \\ &= [\Phi \circ (\mu_1 \circ (\pi_1 \times p_1 \circ \pi_2) \times p_2 \circ \pi_2)]^* \\ &\quad + ((\pi_1 \times p_1 \circ \pi_2)^* \mu_1{}^* \tau)\, [\Phi \circ (\mu_1 \circ (\pi_1 \times p_1 \circ \pi_2) \times p_2 \circ \pi_2)]^* \circ X_1 \\ &\quad + \delta_{j2}\, \tau_1\, \tau_2\, (\mu_1 \circ (\pi_1 \times p_1 \circ \pi_2) \times p_2 \circ \pi_2)^* \widetilde{\partial}_t \Phi^* \\ &\quad + \delta_{j3}\, \tau_1\, (e^{t_2} - 1)(\mu_1 \circ (\pi_1 \times p_1 \circ \pi_2) \times p_2 \circ \pi_2)^* \Phi^* X_1 \\ &= (\Phi \circ (\pi_1 \times \Phi \circ \pi_2))^* + (\tau_1 + \tau_2)\, (\Phi \circ (\pi_1 \times \Phi \circ \pi_2))^* \circ X_1 \\ &\quad + \delta_{j3}\, \tau_1\, (e^{t_2} - 1)(\Phi \circ (\pi_1 \times \Phi \circ \pi_2))^* \circ X_1 \\ &\quad + \delta_{j2}\, \tau_1\, \tau_2\, (\Phi \circ (\pi_1 \times \Phi \circ \pi_2))^* \circ X_0. \end{aligned} \tag{3.4}$$

On the other hand,

$$\begin{aligned} (\pi_1 \times \Gamma \circ \pi_2)^* \Gamma^* &= (\pi_1 \times \Gamma \circ \pi_2)^* (\Phi^* + (p_1{}^*\tau)(\Phi^* X_1)) \\ &= (\Phi \circ (\pi_1 \times \Gamma \circ \pi_2))^* + \tau_1\, (\Phi \circ (\pi_1 \times \Gamma \circ \pi_2))^* \circ X_1. \end{aligned} \tag{3.5}$$

Thus, everything comes down to compare $\Phi \circ (\pi_1 \times \Gamma \circ \pi_2)$, with $\Phi \circ (\pi_1 \times \Phi \circ \pi_2)$. Now we claim that

$$(\Phi \circ (\pi_1 \times \Gamma \circ \pi_2))^* = (\Phi \circ (\pi_1 \times \Phi \circ \pi_2))^* + \tau_2 (\pi_1 \times \Phi_\alpha{}^{-1} \circ \pi_2)^* \overline{X}_1 \Phi^*. \tag{3.6}$$

To prove this we may again assume that $\Phi^* f = \sum_i p_1{}^* f_i p_2{}^* g_i$. (By completeness of the product sheaf on $\mathcal{V}$, if the result is true for this particular form of $\Phi^* f$, the result will be true in general). Now

$$\begin{aligned}(\Phi \circ (\pi_1 \times \Gamma \circ \pi_2))^* f &= \sum_i (\pi_1 \times \Gamma \circ \pi_2)^* (p_1{}^* f_i p_2{}^* g_i) \\ &= \sum_i \pi_1{}^* f_i ((\Phi \circ \pi_2)^* g_i + \tau_2 (\Phi \circ \pi_2)^* X_1 g_i) \\ &= (\pi_1 \times \Phi \circ \pi_2)^* (\sum_i p_1{}^* f_i p_2{}^* g_i) + \tau_2 \Big(\sum_i \pi_1{}^* \overline{f}_i (\Phi \circ \pi_2)^* X_1 g_i\Big),\end{aligned}$$

where we have put, $\overline{f}_i = (f_i)_0 - (f_i)_1 \tau$, because we permuted places with the odd variable τ_2. Note that

$$\pi_2{}^* \Phi^* X_1 g_i = \pi_2{}^* \Phi_\alpha{}^{-1*} \overline{X}_1 p_2{}^* g_i,$$

because

$$\begin{aligned}\Phi_\alpha{}^* \Phi^* X_1 g_i &= p_2{}^* X_1 g_i = \overline{X}_1 p_2{}^* g_i \\ &\rightarrow \Phi^* X_1 g_i = (\Phi_\alpha{}^{-1})^* \overline{X}_1 p_2{}^* g_i \\ &\rightarrow \pi_2{}^* \Phi^* X_1 g_i = (\Phi_\alpha{}^{-1} \circ \pi_2)^* \overline{X}_1 p_2{}^* g_i.\end{aligned}$$

Therefore,

$$\begin{aligned}(\Phi \circ (\pi_1 \times \Gamma \circ \pi_2))^* f &= \Big(\Phi \circ (\pi_1 \times \Phi \circ \pi_2)\Big)^* f \\ &\quad + \tau_2 \Big((\pi_1 \times \Phi_\alpha{}^{-1} \circ \pi_2)^* \sum_i (p_1{}^* \overline{f}_i \overline{X}_1 p_2{}^* g_i)\Big) \\ &= \Big(\Phi \circ (\pi_1 \times \Phi \circ \pi_2)\Big)^* f \\ &\quad + \tau_2 \Big((\pi_1 \times \Phi_\alpha{}^{-1} \circ \pi_2)^* \sum_i (\overline{X}_1 p_1{}^* f_i\, p_2{}^* g_i)\Big),\end{aligned}$$

where we have used the fact that X_1 is an odd derivation to revert from $\overline{f}_i$ to f_i. Thus eq. (3.6) above is true. In particular, the right hand side of eq. (3.5) is,

$$\begin{aligned}\text{RHS of (3.5)} = {} & (\Phi \circ (\pi_1 \times \Phi \circ \pi_2))^* + \tau_2 (\pi_1 \times \Phi_\alpha{}^{-1} \circ \pi_2)^* \overline{X}_1 \Phi^* \\ & \tau_1 \Big((\Phi \circ (\pi_1 \times \Phi \circ \pi_2))^* X_1 + \tau_2 (\pi_1 \times \Phi_\alpha{}^{-1} \circ \pi_2)^* \overline{X}_1 \Phi^* X_1\Big).\end{aligned}$$

If we now compare this expression with the right hand side of eq. (3.4), we conclude that the diagram (3.3) commutes, if and only if

$$(\pi_1 \times \Phi_\alpha{}^{-1} \circ \pi_2)^* \overline{X}_1 \Phi^* = (1 + \delta_{j3} (e^{t_2} - 1))(\Phi \circ (\pi_1 \times \Phi \circ \pi_2))^* X_1,$$

and,

$$(\pi_1 \times \Phi_\alpha{}^{-1} \circ \pi_2)^* \overline{X}_1 \Phi^* X_1 = \delta_{j2} (\Phi \circ (\pi_1 \times \Phi \circ \pi_2))^* \circ X_0.$$

Equivalently, if and only if

$$\begin{aligned} \overline{X}_1 \circ \Phi^* &= (1 + \delta_{j,3} (e^{-t_2} - 1)) \Phi^* \circ X_1 \quad \text{and} \\ \overline{X}_1 \circ \Phi^* \circ X_1 &= \delta_{j2} \Phi^* \circ X_0, \end{aligned} \tag{3.7}$$

where we have used the fact that $\pi_1 \times \Phi_\alpha \circ \pi_2$ is invertible, and its inverse is $\pi_1 \times {\Phi_\alpha}^{-1} \circ \pi_2$, and the fact that

$$\Big(\Phi \circ (\pi_1 \times \Phi \circ \pi_2)\Big) \circ (\pi_1 \times {\Phi_\alpha}^{-1} \circ \pi_2) = \Phi,$$

(both assertions are easy to check). Note the appearance of e^{-t_2} in the right hand side of the first equation. This follows from, $(\pi_1 \times \Phi_\alpha \circ \pi_2)^* e^{(p_1 \circ \pi_2)^* t} = e^{(\alpha \circ p_1 \circ \pi_2)^* t}$, which is in turn a consequence of the definition of Φ_α. In particular, it follows from these equations that

$$\delta_{j2} \Phi^* \circ X_0 = (1 + \delta_{j,3} (e^{-t_2} - 1)) \Phi^* \circ X_1 \circ X_1.$$

For j = 1, and 3 this equation says that $\Phi^* \circ X_1 \circ X_1 = 0$, and since Φ^* is monic, $[X_1, X_1] = 2 X_1 \circ X_1 = 0$. For j = 2, the same equation says $\Phi^* (2 X_0 - [X_1, X_1]) = 0$.
On the other hand, for j = 1, and 2, the first equation in (3.7) says that $\overline{X}_1 \circ \Phi^* = \Phi^* \circ X_1$. Applying $\widetilde{\partial}_t$ on both sides, and using both, proposition 3.2, and the statement (3.4) in the proof of lemma 3.3, we get

$$\overline{X}_1 \circ \Phi^* \circ X_0 = \Phi^* \circ X_0 \circ X_1.$$

Now the original equation may be used again in the left hand side and replace $\overline{X}_1 \circ \Phi^*$ by $\Phi^* \circ X_1$, to finally obtain, $\Phi^*([X_0, X_1]) = 0$. For j = 3 the procedure is exactly the same: Apply $\widetilde{\partial}_t$ on both sides, then use proposition 3.2, the statement (3.4) in the proof of lemma 3.3, and finally equation (3.7) again to substitute the value of $\overline{X}_1 \circ \Phi^*$. □]

Example 3.7.

Let $\Omega(M)$ be the sheaf of differentiable forms on the differentiable manifold M of dimension m. The pair $(M, \Omega(M))$ is a supermanifold of dimension (m, m). Some distinguished supervector fields are:

(i) The Lie derivative, $\mathcal{L}_X$, with respect a vector field X in M,

(ii) The contraction, i_X, with respect a vector field X, and,

(iii) The exterior derivative, d.

Now, the integral flow of $\mathcal{L}_X$ is the pull-back, Φ^*, of the integral flow of X. The integral flow of i_X is given by the map $\Gamma^* = id^* + \tau\, i_X$. Finally, the integral flow of d is given by the map $\Gamma^* = id^* + \tau\, \mathrm{d}$. It is easy to construct a supervector field that does not define any type of $\mathbb{R}^{1|1}$ action. For example, $X_1 = \mathrm{d} + i_X$. This is so because $[X_1, X_1] = 2\mathcal{L}_X$. The integral flow of X_1 is given by the map $\Gamma^* = id^* + \tau\, (\mathrm{d} + i_X)$. The reader can also check directly that the integral flows of d and i_X do not commute.

An example of supervector field defining a Type 2 $\mathbb{R}^{1|1}$ action is given by the derivation, $i_X + \mathrm{d} + 2\mathcal{L}_X$. Its integral flow is $\Gamma^* = \Phi^* \circ (id^* + \tau(\mathrm{d} + i_X))$, where Φ^* is the integral flow of the vector field $2X$.

For an example of a supervector field defining a Type 3 $\mathbb{R}^{1|1}$ action let Id be the identity map of the cotangent bundle. Id can be viewed as a vector-valued differential form of degree 1. The contraction of this form with differential forms defines an algebraic derivation of degree 0; the latter shall be denoted by i_{Id}. Note that if $\beta_{(p)}$ is a differential form of degree p then $i_{\mathrm{Id}}(\beta_{(p)}) = p\beta_{(p)}$. The integral flow of i_{Id} is given by $\Phi^* \beta_{(p)} = \mathrm{e}^{pt}\beta_{(p)}$. Now, consider the derivation $D = i_{\mathrm{Id}} + i_X$. This is of type 3 because $[i_X, i_{\mathrm{Id}}] = i_X$. The associated derivation ${\Phi_\alpha}^{-1^*} \circ \overline{X}_1 \circ {\Phi_\alpha}^*$ defined in lemma 3.3 is just $\mathrm{e}^{-t} i_X$. By theorem 3.5, the integral flow of D is given by $\Gamma^* \beta_{(p)} = \mathrm{e}^{pt}(\beta_{(p)} + \tau\,\mathrm{e}^{-t} i_X \beta_{(p)})$.

Our next result states precisely under what conditions the integral flows of two supervector fields commute. In order to keep the notation simple, the proof is given only for complete supervector fields. The general case considers the intersection of the domains of the flows and it is handled similarly. Note that in the C^∞ category, the statement that the flows ϕ and ψ (of $\widetilde{X}_0$, and $\widetilde{Y}_0$, resp.) commute is that *for all* t_1, *and all* t_2,

$$\phi_{t_1} \circ \psi_{t_2} = \psi_{t_2} \circ \phi_{t_1}.$$

Thus, the statement for the $\mathbb{Z}_2$-graded category has to use the *twist morphism*, $T\colon \mathbb{R}^{1|1} \times \mathbb{R}^{1|1} \to \mathbb{R}^{1|1} \times \mathbb{R}^{1|1}$, defined by the conditions

$$\rho_1{}^* T^* = \rho_2{}^* \quad \text{and} \quad \rho_2{}^* T^* = \rho_1{}^*,$$

where, ρ_1, and ρ_2 are the projections of the product $\mathbb{R}^{1|1} \times \mathbb{R}^{1|1}$.

Proposition 3.8. *Let* $X = X_0 + X_1$, *and* $Y = Y_0 + Y_1$ *be supervector fields on* $\mathcal{M} = (M, \mathcal{A}_M)$, *and let* Γ, *and* Θ *be their corresponding integral flows. Then,* Γ, *and* Θ *commute, i.e.,*

$$\begin{array}{ccc} \mathbb{R}^{1|1} \times \mathbb{R}^{1|1} \times \mathcal{M} & \xrightarrow{(\pi_1 \times \Gamma \circ \pi_2)\circ(T \circ (\pi_1 \times \circ p_1 \circ \pi_2) \times p_2 \circ \pi_2)} & \mathbb{R}^{1|1} \times \mathcal{M} \\ {\scriptstyle \pi_1 \times \Theta \circ \pi_2}\big\downarrow & & \big\downarrow{\scriptstyle \Theta} \\ \mathbb{R}^{1|1} \times \mathcal{M} & \xrightarrow{\quad\Gamma\quad} & \mathcal{M} \end{array}$$

commutes, if and only if

$$[X_0, Y] = 0 \quad \textit{and} \quad [X_1, Y] = 0.$$

Proof. The methods of the proof are the same as those of theorem 3.6. We shall write, $\Gamma^* = \Phi^* + \tau_1 \Phi^* X_1$, and $\Theta^* = \Psi^* + \tau_1 \Psi^* Y_1$, where both, Φ, and Ψ—being the flows of X_0, and Y_0, respectively—define Type 1 supergroup actions of $\mathbb{R}^{1|1}$ in $\mathcal{M}$. Thus, on the one hand one finds (cf. eq. (3.6) in thm. 3.6),

$$(\pi_1 \times \Theta \circ \pi_2)^* \Gamma^* = (\Phi \circ (\pi_1 \times \Psi \circ \pi_2))^* + \tau_2 (\pi_1 \times {\Psi_\alpha}^{-1} \circ \pi_2)^* \overline{Y}_1 \Phi^* + \tau_1 ((\Phi \circ (\pi_1 \times \Psi \circ \pi_2))^* + \tau_2 (\pi_1 \times {\Psi_\alpha}^{-1} \circ \pi_2)^* \overline{Y}_1 \Phi^*) X_1.$$

On the other hand,

$$(\pi_1 \times \Gamma \circ \pi_2)^* \Theta^* = (\Psi \circ (\pi_1 \times \Phi \circ \pi_2))^* + \tau_2 (\pi_1 \times {\Phi_\alpha}^{-1} \circ \pi_2)^* \overline{X}_1 \Psi^* + \tau_1 ((\Psi \circ (\pi_1 \times \Phi \circ \pi_2))^* + \tau_2 (\pi_1 \times {\Phi_\alpha}^{-1} \circ \pi_2)^* \overline{X}_1 \Psi^*) Y_1.$$

Thus, acting on both sides of this equation with $(T \circ (\pi_1 \times \circ p_1 \circ \pi_2) \times p_2 \circ \pi_2)^*$, we simply get τ_1, and τ_2 interchanged. Thus, the diagram in the statement commutes, if and only if, the following equations are satisfied:

$$\begin{aligned} \Phi \circ (\pi_1 \times \Psi \circ \pi_2) &= \Psi \circ (\pi_1 \times \Phi \circ \pi_2), \\ (\pi_1 \times {\Psi_\alpha}^{-1} \circ \pi_2)^* \overline{Y}_1 \Phi^* &= (\Psi \circ (\pi_1 \times \Phi \circ \pi_2))^* Y_1, \\ (\Phi \circ (\pi_1 \times \Psi \circ \pi_2))^* X_1 &= (\pi_1 \times {\Phi_\alpha}^{-1} \circ \pi_2)^* \overline{X}_1 \Psi^*, \\ (\pi_1 \times {\Psi_\alpha}^{-1} \circ \pi_2)^* \overline{Y}_1 \Phi^* X_1 &= -(\pi_1 \times {\Phi_\alpha}^{-1} \circ \pi_2)^* \overline{X}_1 \Psi^* Y_1, \end{aligned}$$

where the minus sign in the last equation is the result of writing $\tau_2\tau_1 = -\tau_1\tau_2$. We now substitute the first equation in the second and third, and the resulting three in the last one. We finally act on such equations from the left with, $(\pi_1 \times \Psi_\alpha \circ \pi_2)^*$, and $(\pi_1 \times \Psi_\alpha \circ \pi_2)^*$, appropriately, and end up with the following system of equivalent equations:

$$\begin{aligned} \Phi \circ (\pi_1 \times \Psi \circ \pi_2) &= \Psi \circ (\pi_1 \times \Phi \circ \pi_2), & X_1 \circ Y_1 &= -Y_1 \circ X_1, \\ \Phi^* \circ Y_1 &= \overline{Y}_1 \circ \Phi^*, & \Psi^* \circ X_1 &= \overline{X}_1 \circ \Psi^*, \end{aligned}$$

and these equations hold true if and only if

$$\begin{aligned} [X_0, Y_0] &= 0, & [X_1, Y_1] &= 0, \\ [X_0, Y_1] &= 0, & [X_1, Y_0] &= 0. \end{aligned}$$

(Note that these equations are not equivalent to $[X, Y] = 0$). □

Remark. In the $\mathcal{C}^\infty$-category there exists a bijection between the set $\operatorname{Der}\mathcal{C}^\infty(M)$, of complete vector fields on a smooth manifold M, and the subset

$$\operatorname{Hom}(\mathcal{C}^\infty(M), \mathcal{C}^\infty(\mathbb{R} \times M))^{\mathbb{R}},$$

of algebra maps $\Phi^*: \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(\mathbb{R} \times M)$, satisfying

(i) ${\Phi_0}^* = \mathrm{id}$, and,

(ii) ${\Phi_{t_1}}^* \circ {\Phi_{t_2}}^* = {\Phi_{t_1+t_2}}^*$, for all t_1, and t_2 in $\mathbb{R}$,

where ${\Phi_t}^* = \mathrm{ev}|_t \circ \Phi^* \in \operatorname{Aut}\mathcal{C}^\infty(M)$, for each $t \in \mathbb{R}$.

In order to find a similar characterization in the $\mathbb{Z}_2$-graded category, note first that if $\Gamma \in \mathrm{Hom}(\mathcal{A}_M(M), \mathcal{A}_{\mathbb{R}^{1|1}\times\mathcal{M}}(\mathbb{R}\times M))^{\mathbb{R}^{1|1}}$ (where $\mathcal{A}_{\mathbb{R}^{1|1}\times\mathcal{M}}$ denotes the structure sheaf of $\mathbb{R}^{1|1} \times (M, \mathcal{A}_M)$), then,

$$\Gamma^* f = \Phi^* f + \tau Z_1 f,$$

for any $f \in \mathcal{A}_M(M)$. It is easy to verify that Z_1 must be an odd $\mathbb{R}$-linear map $Z_1\colon \mathcal{A}_M(M) \to \mathcal{A}_{\mathbb{R}^{1|0}\times\mathcal{M}}(\mathbb{R}\times M)$, satisfying

$$Z_1(fg) = Z_1(f)\,\Phi^*(g) + (-1)^{|f|}\Phi^*(f)\,Z_1(g),$$

for all f, and g in $\mathcal{A}_M(M)$, and furthermore, that
$\Phi \in \mathrm{Hom}(\mathcal{A}_M(M), \mathcal{A}_{\mathbb{R}^{1|0}\times\mathcal{M}}(\mathbb{R}\times M))$. Our last result in this section says that there is a similar correspondence in the $\mathbb{Z}_2$-graded category. The statement and its proof are simple rephrasings of the proof of theorem 3.5, and the previous lemmas. (We are indebted to Prof. J. Muñoz Masqué, for bringing this point to our attention).

Proposition 3.9. *Let $\mathcal{M} = (M, \mathcal{A}_M)$ be a supermanifold, and let $\mathcal{A}_{\mathbb{R}^{1|1}\times\mathcal{M}}$, and $\mathcal{A}_{\mathbb{R}^{1|0}\times\mathcal{M}}$, be the structure sheaves of the supermanifolds $\mathbb{R}^{1|1}\times\mathcal{M}$, and $\mathbb{R}^{1|0}\times\mathcal{M}$, respectively. There exists a one-to-one correspondence between the set* Der $\mathcal{A}_M(M)$, *of complete supervector fields on $(M, \mathcal{A}_M)$, and the subset,*

$$\mathrm{Hom}(\mathcal{A}_M(M), \mathcal{A}_{\mathbb{R}^{1|1}\times\mathcal{M}}(\mathbb{R}\times M))^{\mathbb{R}^{1|1}},$$

of superalgebra maps Γ^*, such that

(i) $\Gamma_0^* = ev|_{t=0} \circ \Gamma^* = \mathrm{id}.$,

(ii) The homomorphism $\Phi^* \in \mathrm{Hom}(\mathcal{A}_M(M), \mathcal{A}_{\mathbb{R}^{1|0}\times\mathcal{M}}(\mathbb{R}\times M))$ associated to Γ, defines an $\mathbb{R}$-action on $\mathcal{M}$, and naturally extends to an $\mathbb{R}^{1|1}$-action.

(iii) The odd $\mathbb{R}$-linear map $Z_1\colon \mathcal{A}_M(M) \to \mathcal{A}_{\mathbb{R}^{1|0}\times\mathcal{M}}(\mathbb{R}\times M)$ associated to Γ is such that

$$\Phi^* \circ \overline{Z}_1 \in p_2{}^* \,\mathrm{Der}\, \mathcal{A}_M(M);$$

that is, $\Phi^* \circ \overline{Z}_1$ comes from an odd supervector field on $(M, \mathcal{A}_M)$.

(Note that the third condition means that the map $\Phi^* \circ X_1$, which in principle is just a derivation from $\mathcal{A}_M$ into $\mathcal{A}_{\mathbb{R}^{1|0}\times\mathcal{M}}$, actually defines an odd supervector field on $(M, \mathcal{A}_M)$).

4. Integral flows and Lie superderivatives of superforms

This section is included for the sake of completeness. We shall proceed along the lines of refs. [10] and [12]. Our aim is to define the Lie derivative of *any* superform, ω, on a given superdomain, with respect to *any* supervector field, X. Moreover, we want to relate our definition to the integral flow Γ, of X, and also, be able to prove that the usual algebraic characterization given in terms of interior multiplication and exterior differentiation holds true. Our guiding principle has been the fact that the algebraic formula for Lie superderivatives, also called the Cartan formula, $\mathcal{L}_X\,\omega = \mathrm{d} \circ i(X)\,\omega + i(X) \circ \mathrm{d}\,\omega$, makes sense regardless of the peculiarities of the field X (i.e., it is not necessary that its homogeneous components satisfy $[X_0, X_1] = \delta_{j3}\,X_1$, and $[X_1, X_1] = \delta_{j2}\,X_0$). We thus start with the following:

Definition 4.1.
Let X be a supervector field on a superdomain $(M, \mathcal{A}_M)$, and let Γ be its unique integral flow satisfying the initial condition, $\mathrm{ev}|_{t=0}\Gamma^* = \mathrm{id}^*$. Let ω be any superform on $(M, \mathcal{A}_M)$. *The Lie superderivative* of ω is the superform, $\mathcal{L}_X\omega$, given by

$$\mathcal{L}_X\omega = \mathrm{ev}|_{t=0} \circ \widetilde{D} \circ \Gamma^*\omega.$$

Proposition 4.2. *The usual relationship between Lie derivatives on forms, exterior differentiation, and interior multiplication, holds true in the theory of supermanifolds; namely,*

$$\mathcal{L}_X\,\omega = \mathrm{d}\,i(X)\,\omega + i(X)\,\mathrm{d}\omega.$$

Proof. It suffices to verify that both sides yield the same answer when $\omega = f$, and when $\omega = \mathrm{d}f$, for any superfunction f. Now, for $\omega = f$, we have,

$$\begin{aligned}\mathcal{L}_X f &= \mathrm{ev}|_{t=0} \circ \widetilde{D} \circ \Gamma^* f \\ &= \mathrm{ev}|_{t=0} \circ \Gamma^* \circ X f \\ &= X f = i(X)\,\mathrm{d}f,\end{aligned}$$

where use has been made of the differential equation for the flow of X, the initial condition, and the definition (as in ref. [3]) of the exterior derivative on superfunctions.

Let us now assume that $\omega = \mathrm{d}f$. Let d, and $\mathrm{d}_{\mathbb{R}}$, be the exterior differentiation operators on the supermanifolds $(M, \mathcal{A}_M)$, and $\mathbb{R}^{1|1}$, respectively. Therefore, the exterior differentiation operator on $\mathbb{R}^{1|1} \times (M, \mathcal{A}_M)$ is defined by the conditions

$$\tilde{\mathrm{d}} \circ \pi_1^* = \pi_1^* \circ \mathrm{d}_{\mathbb{R}} \quad \text{and} \quad \tilde{\mathrm{d}} \circ \pi_2^* = \pi_2^* \circ \mathrm{d}.$$

"Γ^* commutes with d", then means that $\tilde{\mathrm{d}} \circ \Gamma^* = \Gamma^* \circ \mathrm{d}$. Moreover, the operators $\tilde{\mathrm{d}}$, and $\widetilde{D}$ commute with each other, as can be checked directly from the definitions. With these preliminaries in mind, one now has the following:

$$\begin{aligned}\mathcal{L}_X\,\mathrm{d}f &= \mathrm{ev}|_{t=0} \circ \widetilde{D} \circ \Gamma^*\,\mathrm{d}f = \mathrm{ev}|_{t=0} \circ \widetilde{D} \circ \tilde{\mathrm{d}}\Gamma^* f \\ &= \mathrm{ev}|_{t=0} \circ \tilde{\mathrm{d}} \circ \widetilde{D} \circ \Gamma^* f = \mathrm{d} \circ \mathrm{ev}|_{t=0} \circ \widetilde{D} \circ \Gamma^* f \\ &= \mathrm{d} \circ \mathrm{ev}|_{t=0} \circ \Gamma^* \circ X f = \mathrm{d}(i(X)\,\mathrm{d}f),\end{aligned}$$

where we have used the superdifferential equation, and the initial condition. □

In the case when the integral flow Γ defines a Lie supergroup action of $\mathbb{R}^{1|1}$ in $(M, \mathcal{A}_M)$ we can say even more:

Proposition 4.3. *Let X be a supervector field satisfying any of the conditions of theorem 3.6, and let Γ be its unique integral flow satisfying the initial condition, $ev|_{t=0}\Gamma^* = \mathrm{id}^*$. Then, for any superform ω,*

$$\Gamma^* \mathcal{L}_X \omega = \widetilde{D} \circ \Gamma^* \omega.$$

Proof. This is a straightforward verification using, $\omega = f$, and $\omega = \mathrm{d}f$, for an arbitrary superfunction f. The only difference with proposition 4.2 above is that Thm. 3.6 now guarantees that the superdifferential equation satisfied by Γ is $\widetilde{D}\Gamma^* = \Gamma^* X$. □

5. Left invariant superfields on $\mathbb{R}^{1|1}$

Following ref. [1], a Lie supergroup is a supermanifold $(G, \mathcal{A}_G)$, with a preferred underlying point, $e \in G$, and two morphisms,

$$\mu\colon (G, \mathcal{A}_G) \times (G, \mathcal{A}_G) \to (G, \mathcal{A}_G) \quad \text{and} \quad \alpha\colon (G, \mathcal{A}_G) \to (G, \mathcal{A}_G),$$

satisfying

(i) $\mu \circ (\mu \circ (p_1 \times p_2) \times p_3) = \mu \circ (p_1 \times \mu \circ (p_2 \times p_3))$,
(ii) $\mu \circ (C_e \times \mathrm{id}) = \mathrm{id} = \mu \circ (\mathrm{id} \times C_e)$,
(iii) $\mu \circ (\alpha \times \mathrm{id}) = C_e = \mu \circ (\mathrm{id} \times \alpha)$.

The left action of $(G, \mathcal{A}_G)$ on itself can be expressed in terms of μ as follows: Let $\{x^j, \theta^\nu\}$ be a local system of coordinates on $(G, \mathcal{A}_G)$. Assume

$$\mu^* f = \mathcal{F}({p_1}^* x^j, {p_2}^* x^j; {p_1}^* \theta^\nu, {p_2}^* \theta^\nu).$$

Then

$$L^* f = \mathcal{F}({x_1}^j, x^j; {\theta_1}^\nu, \theta^\nu),$$

with

$$L^* {x_1}^j = {x_1}^j \quad \text{and} \quad L^* {\theta_1}^\nu = {\theta_1}^\nu.$$

This morphism is invertible, and its inverse, L^{-1}, is given in terms of μ, and α as follows: write

$$\{\mu \circ (\alpha \circ p_1 \times p_2)\}^* f = \mathcal{H}({p_1}^* x^j, {p_2}^* x^j; {p_1}^* \theta^\nu, {p_2}^* \theta^\nu).$$

Then

$$L^{-1^*} f = \mathcal{H}({x_1}^j, x^j; {\theta_2}^\nu, \theta^\nu),$$

with

$$L^{-1^*} {x_1}^j = {x_1}^j \quad \text{and} \quad L^{-1^*} {\theta_1}^\nu = {\theta_1}^\nu.$$

We shall illustrate the use of L in the following examples:

Example 5.1. Let $\mathbb{R}^{1|1}$ be considered with its multiplicative structure; in terms of the local coordinates $\{x, \theta\}$,

$$\mu^* x = p_1{}^* x \, p_2{}^* x + p_1{}^* \theta \, p_2{}^* \theta \quad \text{and} \quad \mu^* \theta = p_1{}^* x \, p_2{}^* \theta + p_1{}^* \theta \, p_2{}^* x.$$

Let GL(1|1) be $\mathbb{R}^{1|1}$ with the point $x = 0$ removed, and structure sheaf the restriction of that of $\mathbb{R}^{1|1}$ to $\mathbb{R} - \{0\}$. Then, GL(1|1) is a Lie supergroup (cf. ref. [1]). The inversion morphism is given by

$$\alpha^* x = \frac{1}{x} \quad \text{and} \quad \alpha^* \theta = -\frac{1}{x^2}\theta.$$

The morphism L referred to above is given by

$$L^* x = x_1 x + \theta_1 \theta \quad \text{and} \quad L^* \theta = x_1 \theta + \theta_1 x,$$

and its inverse is

$$L^{-1^*} x = \frac{1}{x_1}\left(x - \frac{\theta_1}{x_1}\theta\right) \quad \text{and} \quad L^{-1^*} \theta = \frac{1}{x_1}\left(\theta - \frac{\theta_1}{x_1} x\right).$$

Let $X = (f_0 + f_1 \theta)\,\partial_x + (g_0 + g_1 \theta)\,\partial_\theta$, be an arbitrary supervector field on $(\mathbb{R}^{1|1})^*$. Then, X is left invariant, if and only if

$$X = L^* X L^{-1^*} = (L^* X L^{-1^*} x)\,\partial_x + (L^* X L^{-1^*} \theta)\,\partial_\theta.$$

It is then easy to check that X is left invariant, if and only if it is of the form

$$X = \lambda_0\,(x\partial_x + \theta\partial_\theta) + \lambda_1\,(x\partial_\theta - \theta\partial_x),$$

with λ_0, and λ_1 real constants. Note that in this case (assuming $\lambda_0\lambda_1 \neq 0$),

$$[X_0, X_1] = 0 \quad \text{and} \quad [X_1, X_1] = -2\frac{\lambda_1{}^2}{\lambda_0} X_0.$$

The integral flow of X is found as follows: First, it is easy to check that the map,

$$\Phi^* x = \mathrm{e}^{\lambda_0 t}\, x, \quad \Phi^* \theta = \mathrm{e}^{\lambda_0 t}\, \theta,$$

is the integral flow of the even part X_0, satisfying $\partial_\tau \Phi^* = 0$ (i.e., as if $X_1 = 0$). As we have seen, the integral flow of $X_0 + X_1$ is given by $\Gamma^* = \Phi^* + \tau\,\Phi^* \circ X_1$. It is then easy to check that

$$\Gamma^* x = \mathrm{e}^{\lambda_0 t}\,(x - \lambda_1 \tau\,\theta), \quad \Gamma^* \theta = \mathrm{e}^{\lambda_0 t}\,(\theta + \lambda_1 \tau\, x).$$

Example 5.2.

Now consider $\mathbb{R}^{1|1}$ with the Type 2 Lie supergroup structure: In terms of the local coordinates $\{x, \theta\}$,

$$\mu_2{}^* x = p_1{}^* x + p_2{}^* x + p_1{}^* \theta \, p_2{}^* \theta \quad \text{and} \quad \mu_2{}^* \theta = p_1{}^* \theta + p_2{}^* \theta.$$

The left multiplication morphism L is,

$$L^* x = x_1 + x + \theta_1 \theta \quad \text{and} \quad L^* \theta = \theta_1 + \theta,$$

and its inverse is,

$$L^{-1^*}x = -x_1 + x - \theta_1\theta \quad \text{and} \quad L^{-1^*}\theta = -\theta_1 + \theta.$$

Let $X = (f_0 + f_1\theta)\,\partial_x + (g_0 + g_1\theta)\,\partial_\theta$, be a supervector field on $\mathbb{R}^{1|1}$. Then, the condition, $X = L^*XL^{-1^*}$, for left invariance, leads to the following: *X is left invariant, if and only if it is of the form,*

$$X = \lambda_0\,\partial_x + \lambda_1\,(\partial_\theta + \theta\partial_x),$$

with λ_0, and λ_1 real constants. Note that, if $\lambda_0\lambda_1 \neq 0$,

$$[X_0, X_1] = 0 \quad \text{and} \quad [X_1, X_1] = 2\frac{\lambda_1{}^2}{\lambda_0}X_0.$$

The integral flow of X is found as in the previous example and is given by

$$\Gamma^*x = x + \lambda_0 t + \lambda_1\tau\theta \quad \text{and} \quad \Gamma^*\theta = \theta + \lambda_1\tau.$$

Remark. It is interesting to note that there is a Lie supergroup homomorphism (local isomorphism) between the Type 2 supergroup structure of $\mathbb{R}^{1|1}$, and the supergroup $\mathrm{GL}(1|1)$ of Example 5.1 above. This is the map Ψ: $\mathbb{R}^{1|1} \to \mathrm{GL}(1|1)$, given in terms of local coordinates $\{x, \theta\}$ of $\mathbb{R}^{1|1}$, and $\{y, \xi\}$ of $\mathrm{GL}(1|1)$ by

$$\Psi^*y = \mathrm{e}^x \quad \text{and} \quad \Psi^*\xi = \mathrm{e}^x\theta.$$

It is a straightforward matter to check that this is the unique (locally invertible) morphism satisfying

$$\mu \circ (\Psi \circ p_1 \times \Psi \circ p_2) = \Psi \circ \mu_2,$$

where μ and μ_2 are as in the previous examples.

Example 5.3.

Finally consider $\mathbb{R}^{1|1}$ with the Type 3 Lie supergroup structure: In terms of the local coordinates $\{x, \theta\}$,

$$\mu_3{}^*x = p_1{}^*x + p_2{}^*x \quad \text{and} \quad \mu_3{}^*\theta = \mathrm{e}^{p_2{}^*x}p_1{}^*\theta + p_2{}^*\theta.$$

The left multiplication morphism L is,

$$L^*x = x_1 + x \quad \text{and} \quad L^*\theta = \mathrm{e}^x\theta_1 + \theta,$$

and its inverse is,

$$L^{-1^*}x = -x_1 + x \quad \text{and} \quad L^{-1^*}\theta = -\mathrm{e}^{x-x_1}\theta_1 + \theta.$$

Let $X = (f_0 + f_1\theta)\,\partial_x + (g_0 + g_1\theta)\,\partial_\theta$, be a supervector field on $\mathbb{R}^{1|1}$. The condition, $X = L^*XL^{-1^*}$, leads this time to the following: *X is left invariant, if and only if it is of the form,*

$$X = \lambda_0\,(\partial_x + \theta\,\partial_\theta) + \lambda_1\,\partial_\theta,$$

with λ_0, and λ_1 real constants. Note that

$$[X_0, X_1] = -\lambda_0 X_1 \quad \text{and} \quad [X_1, X_1] = 0.$$

The integral flow of X is found as in the previous examples, and is given by

$$\Gamma^* x = x + \lambda_0 t \quad \text{and} \quad \Gamma^* \theta = \mathrm{e}^{\lambda_0 t} \theta + \lambda_1 \tau.$$

In all these examples, the integral flows of the left invariant supervector fields under consideration do not define Type 1 $\mathbb{R}^{1|1}$-actions on the supergroups they are respectively defined. In fact, $[X_1, X_1] \neq 0$ in 5.1-5.2, and $[X_0, X_1] \neq 0$ in 5.3 (See Thm. 3.6). On the other hand, the integral flows in 5.2, and 5.3, trivially recover the multiplication map μ_j of proposition 2.1 for $\lambda_0 = 1 = \lambda_1$.

Remark.

Let $\operatorname{Der} \mathcal{A}_G(G)^{\mathcal{A}_G(G)}$ be the left invariant derivations on $(G, \mathcal{A}_G)$. Note that the Lie supergroup structures we are dealing with here do not satisfy the properties stated in ref. [3]; namely, that the function and exterior factors can be recovered from the left invariant supervector fields. In particular, it is not true that $\mathcal{C}^\infty(G)$ is isomorphic to

$$\mathcal{C}_0(G) = \{f \in \mathcal{A}_G(G) \mid Xf = 0, \text{ for all odd } X \in \operatorname{Der} \mathcal{A}_G(G)^{\mathcal{A}_G(G)}\}.$$

In both examples above we obtain

$$\mathcal{C}_0(G) = \{f \in \mathcal{A}_G(G) \mid f = c_0;\ c_0 \text{ constant}\} \simeq \mathbb{R}.$$

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